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Thesis Title: Practical Stability Analysis of a Three-Dimensional Storm

Date of completion of requirements for award: 19 March 2013

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Practical Stability Analysis of a Three-Dimensional Swarm

A Master of Science thesis presented
by
Jai Raj
to
The School of Computing, Information and Mathematical Sciences
in partial fulfillment of the requirements
for the degree of
Master of Science
in the subject of
Applied Mathematics

The University of the South Pacific
Fiji Islands
December 2012
Declaration of Originality

I hereby declare that this thesis is original to the best of my knowledge; citations and references have been acknowledged and the main result formulated has not been previously submitted for a university degree either in whole or in part elsewhere.

Jai Raj  
December, 2012.

Declaration by Supervisors

This to certify that the Master’s Thesis entitled “Practical Stability Analysis of a Three-Dimensional Swarm” by Mr. Jai Raj, under our direct supervision represents the original research work of the candidate.

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(Co-supervisor)
PRACTICAL STABILITY ANALYSIS
OF A THREE-DIMENSIONAL SWARM

Abstract

Swarm navigation is an active research domain for the last couple of decades. It can be considered as a motion planning and control problem when contextualized with mobile robots, whether it be swarms or vehicular agents. The capability of biological systems to autonomously maneuver, track and pursue evasive targets in a cluttered 3D environment is vastly superior to any engineered system. It is considered an emergent behavior arising from simple rules that are followed by individuals and may not involve any central coordination.

This research thesis considers a swarm of boids, which is composed of a spherical-shape. The challenges that one may face in getting the desired emergent behaviors are the complexity of the control algorithms, intensive computations and the difficulty of computer simulations, given the constraints arising from the centering of the swarm of boids. Thus, motion planning and control of swarm of boids is an interesting yet complicated multi-tasking problem.

The introductory chapter describes how the swarming behavior and pattern has been modelled and analyzed by researchers. It also outlines some of the milestone work done on the field of swarming with emphasis coming from the various sectors of research such biology, physics and engineering.

While the literature in inundated with numerous algorithms that address the motion planning and control problem of mobile robots, this research utilizes a relatively established artificial potential fields method, known as the Lyapunov-based control scheme. The control scheme provides a simple but effective means of constructing continuous control laws for dynamical systems. In parallel, a system may be unstable yet the system may oscillate sufficiently near this state that its performance is acceptable, basically practically stability will be considered.

The first part and the second part of the thesis uses, via the Lyapunov-based control scheme to address the various sub-tasks involved in kinodynamic planning and control problem. Each of the simulations in the two chapters are separate projects and has its own unique strengths and contributions. The swarms exhibit emergent behaviors which arises by varying the control and convergence parameters.

The final part is arguably the main strength and the application of the thesis. Here, we introduce vehicular agents that move from an initial to a final state whilst avoiding the fixed
obstacles and the swarm of boids within the vicinity of the workspace. This is the first time swarms are introduced together with vehicular agents in environments cluttered with fixed obstacles within an artificial potential field’s method framework. The controllers also guarantee the execution of an array of sub-tasks, which include: goal convergence for the vehicular agents, team coordination and cohesion, and adherence to nonholonomic and kinodynamic constraints.

The effectiveness of the control scheme and the associated velocity-based and acceleration based controllers is demonstrated via computer simulations for various applications and scenarios of the swarms and the vehicular agents respectively, throughout the thesis.
I wish to express my profound gratitude to all the people who have contributed immensely to the successful completion of the thesis.

First and foremost, my profound thanks and the people to whom I am indebted to are my thesis advisors, Dr. Jito Vanualailai and Dr. Bibhya Nand Sharma of the School of Computing, Information and Mathematical Sciences at the University of the South Pacific, for introducing me to the indispensable method of Lyapunov. I deeply express my sincere appreciation for their copious valuable suggestions, motivation, timely encouragement and counselling in every phase of the arduous but meritorious thesis.

Besides, I would like to thank Mr. Shonal Singh for his encouragement and helping hand in the successful completion of my research thesis.

I would also like to thank my colleagues at The University of the South Pacific, especially Mr. Avinesh Prasad, Mr. Varunesh Rao and Mr. Kaylash Chaudhary, for their enthusiastic support and technical assistance.

Lastly, and most importantly, I wish to thank my family members. They have been on my side always, supporting me, inspiring me to keep on working hard, and making sure that I live the day to see my dream come true.
Dedication

This thesis is dedicated
to my parents
Mr. Ind Raj and Mrs. Saiful Nisha.
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Introduction

"He must be a dull man who can examine the exquisite structure of a comb, so beautifully adapted to its end, without enthusiastic admiration."  Charles Robert Darwin (1809 - 1882)

The role of biologically inspired algorithms has inspired researchers since numerous problems can be solved without rigorous mathematical approaches. It is basically the category of algorithms that imitate the way nature performs. This set of algorithms falls under various categories such as Artificial neural networks, Genetic algorithms, Evolutionary algorithms, Particle swarm optimization, Ant colony optimization, Fuzzy logic and others.

Swarming is based on many unsophisticated entities that cooperate in order to exhibit a desired behavior. Inspiration for the design of these desired behavior is taken from the collective behavior of social insects such as ants, termites, bees, and wasps, as well as from the behavior of other animal societies such as flocks of birds or schools of fish, has mesmerized researchers for many years that yearned to understand the underlying cooperative dynamics [99]. Even though the single members of these societies are unsophisticated individuals, they are able to achieve complex tasks in cooperation. Coordinated behavior emerges from relatively simple actions or interactions between the individuals [6]. The swarming behavior is a complex emergent behavior that occurs when individual agents follow simple behavioral rules.

The fact that certain engineering problems can be solved in an ingenious way by roughly mimicking this natural phenomenon [7, 86] has led to greater efforts by mathematicians, engineers, computer scientists, physicists and biologists, in recent years, to seek better understanding of self-organization in organisms, and the formation and the persistence of aggregations [26, 35]. The inspiration of this work was the result of the work done by Okubo in 1980 [99] whereby he introduced and described the various swarm models. Okubo also presented some aspects of the fundamental and preliminary mathematics concepts applied to studying swarms. Since then, the literature has grown substantially, with many significant contributions addressing the mathematical concepts from different perspectives.

The emerging swarm behavior and its principles are now being used by scientists and re-
searchers in many new approaches such as in optimization and in control of robots [24, 103]. The use of robots with the concept of swarming is significantly increasing in the manufacturing arena, not only for productivity enhancement but also for greater versatility and flexibility [44].

Biologically inspired algorithms that mimic the flocking behavior are essential in accomplishing the control objective of a group while ensuring collision-free flight path [114]. Common objectives nowadays include formation flight control, satellite clustering, exploration, surveillance, foraging and cooperate manipulation [127, 126]. The applications of foraging could involve search-and-rescue teams at disaster sites. Teams of robots could be deployed to collect hazardous materials after a spill, nuclear reaction or other accidents in minimal time, hence, saving further loss in the environment. All in all, team(s) of homogeneous (even heterogeneous) robots working towards a common objective can satisfy stringent time, manpower and monetary demands, enhance performance and robustness, and harness desired multi-behaviors, each of which is extremely difficult if not entirely impossible to obtain from single agents [116].

One of the advantages that animals may obtain by grouping is a better chance of avoiding predators. Assuming a predator generally will attack the closest individual, a bird can reduce its "domain of danger", the area in which it can be the closest prey to a predator, by joining a flock. Where there is cover, of course, hiding rather than flocking may be a more effective predator defense strategy.

The swarming behavior and pattern could in fact be modelled and analyzed from different perspectives. Researchers have been considering two important perspectives for the analysis of swarm dynamics [38]. This is a result of the work done by the swarm robotics researchers on behavior based and learning strategies of swarms which lead to the emergence behavior of cooperation in simple tasks. The two fundamentally different approaches are:

(1) The *spatial* approach, and

(2) The *non-spatial* approach.

### 1.1 Spatial Approach

In the spatial approach, the space or the environment is either explicitly or implicitly factored to in the model and the system that is being considered. An explicit space is a fully and clearly defined or formulated space. That is, the swarm is aware of its whereabouts and knows precisely its work space. On the other hand, an implicit space is contained in the nature of something though it is not readily apparent. From the work done on swarm models by [26, 49, 80, 88, 89, 99], it is possible to further categorize the spatial approach for developing a model of a biological swarm into two distinct frameworks, namely:

(1) Eulerian Framework, and
(2) The Lagrangian Framework.

1.1.1 Eulerian Framework

In the Eulerian approach, the swarm is considered a continuum or a continuum model described by its density in one- two or three- dimensional space. The continuum model focuses on the concentration and the population density. In here, the whole swarm is being considered as the reference density of the model [38, 42]. Basically, there is no distinct equation or unique equation for the individual agents. All of the agents are basically governed by one principle equation. The time evolution of animal density is described, in most situations, by partial differential equations [38].

Interactions in the Eulerian models can be classified as either local or non-local. Local interactions [27, 83] represent responses to only immediate members. In contrast, non-local interactions [28, 29, 88, 142, 143] represent responses to neighbors within some region of non-zero radius which leads to integro-differential equations.

The Eulerian Model is a result of both the intrinsic and extrinsic effects [26]. These effects lead to clustering and aggregation. The intrinsic effect results from the direct attraction and repulsion to other members of one’s own kind. The extrinsic effect is exhibited due to some external markers like landmarks which, for instance the bees use for moving and finding food sources, temperature, light, evading predators and the tendency to aggregate. The uneven distribution of resources in the environment leads to clustering. The ecological phenomena of the Eulerian model is known to influence disease growth and transmission rates, predation rates (avoiding predators), selection pressure (survival of the fittest), and evolution of new traits.

The basic equation of the Eulerian model and framework is the \textit{advection-diffusion-reaction} equation of the form

\[ \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left( D(\rho) \frac{\partial \rho}{\partial x} \right) - \frac{\partial}{\partial x} \left( V(\rho) \rho \right) + B(\rho), \]  \hspace{1cm} (1.1)

where the advection term \( \frac{\partial}{\partial x} \left( V(\rho) \rho \right) \) and the diffusion term \( \frac{\partial}{\partial x} \left( D(\rho) \frac{\partial \rho}{\partial x} \right) \) are the joint outcome of the individual behavior and environmental influences, and the reaction term \( B(\rho) \) is due to the population dynamics [38]. In chemistry, earth science and engineering, advection is a transport mechanism of a substance [38]. In mathematics, the advection is the partial differential equation that governs the motion of a conserved scalar field as it is ruled by a known velocity vector field. It is derived using the scalar field’s conservation law, together with Gauss’s theorem, and taking the infinitesimal limit. The derivation of the \textit{advection-diffusion-reaction} equation relies on the principles of superposition.

Moreover, the Eulerian approach is stochastic in nature. A question, posed by Edelstein-
Keshet [26] in her descriptive survey of mathematical models of swarming and social aggregation, vividly elucidates the dichotomy between the two spatial approaches; "are we following a given individual to see how it is affected by its neighbors, or are we watching the herd move past us as a density wave?"

1.1.2 Lagrangian Framework

In the Lagrangian approach, the state (position, the instantaneous velocity and the instantaneous acceleration) of each individual and its relationship with other individuals is studied. It has the benefit of being very intuitive. It is basically an individual-based approach, in which the velocity and the acceleration of a robot can be influenced by spatial coordinates of the individual. In the individual-based models, the primary description is the dynamic motion equation and ordinary differential equation of each individual. Hence it is a natural approach for modelling and analysis of complex social interactions and aggregations [42]. The basis of the Lagrangian framework is that the swarming behavior is a result of an interplay between a long range attraction and a short range repulsion between the individuals in the swarm [26, 41]. This behavior leads to aggregation and formation, which are important for the survival of the members of the swarm in nature [10, 31].

One of the early frameworks within the Lagrangian Framework was carried out by Breder in 1954 [9]. He proposed model which composed of an attraction term and a repulsion term. The terms were inversely proportional to the square of the distance between two individuals. The repulsion was more effective at short distances than the attraction. In a similar manner, at greater distances the attraction was larger than the repulsion [9]. Warburton and Lazarus also investigated and considered the individual-based swarm model by using a family of attractive and repulsive functions to examine the effects of each function on group cohesion [151].

Moreover, the Lagrangian model is a class of computational models for simulating the actions and interactions of autonomous agents, both individual and collective groups with a view to assessing their effects on the system as a whole. It combines elements of game theory, complex systems, emergence, computational sociology, multi-agent systems, and evolutionary programming [94]. The models simulate the simultaneous operations and interactions of multiple agents, in an attempt to re-create and predict the appearance of complex phenomena. The process is one of emergence from the lower level of systems to a higher level. As such, a key notion is that simple behavioral rules generate complex behavior. Individual agents are typically characterized as boundedly rational, presumed to be acting in what they perceive as their own interests, such as reproduction, economic benefit, or social status, using heuristics or simple decision-making rules. In the Lagrangian model, the agents may experience learning, adaptation, and reproduction [8, 38].

Control theory is a branch of engineering and mathematics that deals with the behavior of dynamical systems. The above systems may employ single or any combination of the centralized, decentralized or distributed controllers:
Centralized Control Scheme

In a centralized control system all the decisions are made by the central controller. The decision is based on the feedbacks received from the other components connected to the central controller of the system. In most cases it is easier to develop and analyze a centralized approach.

Sharma et al. [116] has used centralized controllers to mimic biological behavior of flocking and swarming into various forms of simulations. For example, the authors have used the leader follower strategy in [116, 121].

A centralized controller allows high-level control necessary for tasks such as taking seismographic readings or configuring a network of satellite dishes [75]. These tasks demand a precision and directed intentionality which cannot merely emerge from low level interaction. Another advantage of centralized control is that it facilitates human interface. A commander must be able to tell a squadron of planes to abort their mission. If there is centralized control, the human will need only to issue one command. Without centralized control, it will be more difficult, though not impossible, for the command to reach each agent. On the other hand, a totally centralized approach places immense computational demands on the centralized controller and often prohibits real time action [74, 75, 114].

It is easier to implement centralized control using global communications. Completely centralized control architecture is relatively straightforward to implement. But issues arise when dealing with millions of modules such as reaching the limits of bandwidth when using a global communication bus [74]. Lacroix et al. [68] has proposed that several autonomous robots performing a common task is not achievable by a single robot must coordinate their actions in order to succeed. A centralized control strategy may ensure this coordination, but it is often impractical due to strong restrictions on required feedback structure, inter-robot communications and tolerable time-delays.

One of the greatest problems with a centralized approach is that it requires reliable, explicit communication. This communication comes at a great cost. Many small robots simply do not have the capacity to carry sufficiently sophisticated communication devices. Hardware aside, the designer is still faced with the problem of how to synchronize communication. How should the centralized controller handle simultaneous queries from multiple robots? For co-evolutionary learning, diversity is crucial to avoid convergence to locally maximal solutions [114]. Fortunately, many tasks such as gathering environmental information or surveillance have been shown possible when using only simple, local communication. While this capacity can currently be demonstrated only in simulation, it shows that it is possible to create intelligent swarming behavior using only communication with nearest neighbors [116, 121].

Decentralized Control Scheme

In a decentralized controlled system the leader cannot broadcast messages to all the member of a system. In this case all the member of the system will have full knowledge of the scenario/system. The decisions and actions are based on the near neighbor technique [114, 150].
This system is a democratic way of running a formation. A decentralized controller has significant advantages of simple computation and low-cost setup.

The advantage of decentralized control architecture is that computation is shared among members. No single unit needs to do all the heavy computation. This is also thought to be more robust and more easily lends itself to scaling to large numbers of members. It is easier to implement decentralized control using local communications and there are many examples of such [114].

Magana and Tagami [84] proposed that trajectory tracking decentralized adaptive controller for robot manipulators are considerably faster convergence of tracking error, relatively simpler structure, and smoother control activity. Another advantage of this controller is that it only requires local position and velocity measurements, and it does not make use of the exact centralized mathematical model of the robot manipulator. On the other hand it is hard to achieve complex tasks with a completely decentralized architecture that requires only local communication because it is hard to implement behavior in a distributed fashion [84].

**Distributed Control Scheme**

This system deals with grouping of the members in achieving the goal of a system. The communication is inter-group linked. The distributed system can perform different tasks in the various groups of the system without disturbance. In distributed control architectures, failures in other parts of the control system can be compensated by other parts of the system [55, 114].

**1.2 Non-spatial Approach**

In the non-spatial approach, the space, that is the position and the surrounding of the individuals is basically not present in the model. The swarming dynamics are usually described in a non-spatial way in terms of the frequency distributions of the groups of various size [42]. They use the concept of "formation stiffness", a rule necessitating strict observance of the prescribed formation during the motion of the flock. On one end, there are split/rejoin maneuvers [116] which can be required in applications, for example, reconnaissance, sampling and surveillance. On the other hand, there are tight-formations which can be required in applications that require cooperative payload transportation. Then there are low-degree formations, which is required in convoying and demining, that are strict but do allow for slight distortions [116].

The non-spatial approach has a major drawback that limits its ability to describe and analyze the population dynamics. The limitation is that it requires several assumptions about the fusion and fission of groups of various sizes [38].
1.3 The Literature on Swarming by Biologists

The biologists have been working on understanding and trying to model the swarming behavior for a very long time. The literature is inundated with this work and it goes back to the 1920’s. However, major contributions on this work was done by Breder in 1954 [9] and by Okubo in 1980 [99]. Some of the landmark results are outlined below.

Niwa developed a stochastic differential equation model for fish schooling, composed of a locomotory force of attraction/repulsion, and alignment [95]. The analysis was approached from the point of view of fish school as a multi-agent, a self-organizing dynamic structure. Various group properties were investigated as the level of noise relative to interactions was varied. Niwa showed that the group exhibits transitions between behaviors as noise levels change, from amoeba-like, disorganized groups, to polarized groups moving rectilinearly. The fish schools were modelled as an interacting particle system with behavioral and environmental characteristics, such as the gas molecules following the rules of Newton’s mechanic. The equation of motion that was considered from the Lagrangian framework were the mean and the variance of swimming velocity, which illustrated on a feed-forward control structure underlying the self-organization of the school. The work focussed on the transient fluctuation of the centroid velocity. Niwa also showed that by tuning randomness to a particular level, noise can facilitate schooling by reducing the time of onset of schooling [95, 96, 97]. In 2005, Niwa [98], motivated by the fact there is some biological universality in the relationship between school geometry and school biomass of various pelagic fishes in various conditions, he used power-law exponent function to model the increase in the fish school diameter.

In 1991, Warburton and Lazarus [151] investigated and studied the relationship of group cohesion by using mean values and variability in inter-individual distances and group shape. They showed that although stable groups form across most functions with short-range repulsion and long-range attraction, the shape of these response curves affects nearest-neighbor distance, group shape, and level of cohesion. The data showed that the group elongation was positively correlated to the inter-individual distances. Moreover, they showed that the members maintained an equilibrium separation distance, whereby the attraction balances off the repulsion and that the equilibrium distance decreases in larger groups. In addition, the individuals in the group needed to monitor and respond to only a few of its members to elude fragmentation of the groups.

In 1999, Mogilner and Edelstein-Keshet [88] modelled swarming behavior based on non-local interactions. Their model constitutes of the integro-differential equations with convolution terms that describe attraction and repulsion. They demonstrated that if the repulsion is higher than the attraction, then the swarm profile is realistic. This result stipulated the exact behavior as exhibited in biological examples in which the swarm has constant interior density and sharp edges. They also explored the effects of local and non-local density dependent drift and compared their model with some local models and argued that their model accurately represented the swarm behavior.

In 2003, Mogilner et al. [90] improved their results further by using a class of attraction and repulsion functions that are formed using both exponential and power laws. They analytically derived conditions on function parameters to guarantee stable, cohesive well-spaced groups.
They also showed that repulsion must dominate attraction to avoid the group from collapsing to a tight cluster. The result showed that when the number of individuals in a group increases, a separation occurs between the members of the swarm in which the individual distance is preserved versus those in which the physical size of the group is maintained in the expense of greater crowding.

It was proposed by Mogilner et al. [90] in 2003, and subsequently used by Gazi and Passino [39] to establish the stability of certain swarm models. The attraction-repulsion function have an attraction component that dominates for large distances and a repulsion component that dominates for small distances. In 2004, Gazi and Passino [41] extended their 2003 results by also considering interactions between individuals and their environment.

In addition, Gazi and Passino in 2004 [41] provided stability analysis for several cases of the functions considered to characterize swarm cohesiveness, size and ultimate motions while in a cohesive group. They also showed how their new model can be used to accomplish formation control. Specifically, they considered a swarm that is moving in a profile of nutrients or toxic substances, that is, an attraction/repellent profile modelled by quadratic-type or Gaussian-type equations. Their model of emergent behavior of the swarm was a result of the balance between inter-individual interactions and the simultaneous interactions of the swarm members with their environment. By analyzing the stability properties, they provided conditions for collective convergence. Also, in 2004 the authors provided another type of attraction-repulsion function that can be modified to incorporate a finite size of swarm members [40].

The 2003 Gazi and Passino model is isotropic; there is uniformity in attraction or repulsion between all the members of the swarm. Moreover, it is reciprocal; every member $i$ moves toward every other member $j$ exactly the same amount as $j$ moves toward $i$. In 2003, Chu and colleagues generalized the Gazi-Passino model to include anisotropy, or unequal attractive or repulsive forces [15]. Grouping is a property of swarming and this property depends on the attractive or repulsive forces that binds the individual swarms. They illustrated that the members of the swarm aggregate and eventually formed a cohesive cluster of finite size around the swarm center and provided necessary conditions under which the swarm system can be completely stable. Their anisotropic model contains a symmetric coupling matrix that allows the interaction strength between individuals in a swarm to vary. They assumed that the interaction between at least two individuals, but not all, were reciprocal. In 2004, Wang and coworkers removed this reciprocity by adopting an asymmetric coupling matrix [149]. They illustrated the interaction pattern on individual motion in swarm systems. They also studied the aggregation properties of the swarm under the attraction/repellent profile.

A shortcoming of the Gazi-Passino model and its variants mentioned above is that they do not have a collision-avoidance capacity between members of the swarm. The effect of their attraction-repulsion functions is only strong enough for each individual to move towards the center of the swarm and stop without collapsing to a tight cluster. To overcome this shortcoming, Liu and coworkers, in 2005, introduced an unbounded repulsion term which is inversely proportional to the forth-power of the distance between two individuals [81]. The Gaussian-type attractant/repellent nutrient profile in a swarm system was used and it provided insight into the the effect of interaction pattern on self-organized motion. In 2009, they expanded their work to obtain swarm models that are non-reciprocal and exhibit self-organized oscilla-
tions [82]. The function that was proposed and used had infinitely large values of repulsion for vanishing distance between two agents so as to avoid collision. Liu also examined the effect of noise on collective dynamics of the swarm with a white Gaussian noise model.

Furthermore, another shortcoming of the Gazi-Passino-like models is that every individual knows the position of every other individual in the swarm. The models are not scalable; with any increase in swarm size comes greater demand for computing resources, and in applied situations, for sensing capabilities. In 2006, Chen et al. [12, 13] added a component to a Gazi-Passino-like model, and produced an enhanced model that is scalable. Gazi and Passino [41], nonetheless, argued that sensing limitations in engineering applications, like controlling robots, could be solved with technologies such as the Global Positioning System (GPS).

Indeed, distributed control principles have been successively applied to a series of case studies in collective robotics (aggregation and segregation, foraging, collaborative stick pulling, cooperative transportation, flocking and navigation in formation, odor source localization, cooperative mapping, and soccer tournaments) for which several approaches extensively exploit global communication capabilities [86].

Another group of researchers has been working on developing flock models based on the interactions of individuals. Recent result includes those by Olfati [141], who proposed a flock model that allows members of a flock to avoid fixed multiple obstacles. The model displayed that migration of flocks can be performed using a peer-to-peer network of agents, that is, the concept of leaderless swarms. Tanner et al. [139], who used non-smooth analysis to construct a robust flock model; and Gu and Hu [50] who used fuzzy logic and the algebraic graph theory, together with non-smooth analysis, to create functions for collision avoidance between members in a flock. These three scalable models express the three well-known Reynolds’ heuristic flocking rules [109] into precise mathematical statements.

1.4 The Literature on Swarming by Physicists

In parallel to the mathematical biologists, an alternative approach to understanding coherent swarm structures can be traced to the work of physicists. The physicists consider individuals in a swarm as self-propelling particles governed by discrete equations and they study the collective behavior due to their interactions. These models of discrete swarm structures use iterative methods which provide recursion formulas that update the position, velocity and orientation of an individual with respect to other individuals. Extensive simulations are required to validate the models, the simplest of which merely assume that the individuals move at a constant speed, and, at each time step, each one travels in the average direction of motion of those within a local neighborhood. This is either the distributed or the Hamiltonian approach [53]. Researchers all over the world also use the Newton’s Equation of motion to model but not to mention that this is not the only type of model and method under consideration.

A few of the landmark models are discussed in this thesis. Vicsec et al. [148] introduced a model to investigate the emergence of self-ordered motion in a system of particles. The particles are driven with a constant absolute velocity and assumed the average direction of motion of the particles in each time step with some random perturbation. The authors provided
numerical evidence that their model results in a kinematic phase transition. This happens as a result of zero average velocity to finite velocity through spontaneous symmetry breaking of the rotational symmetry.

Czirók et al. [20] studied a non-equilibrium model consisting of self-propelled particles. In this model, the particles move on a plane with a velocity of constant magnitude. At each time step a velocity direction equal to the average direction of the neighbors is chosen by locally interacting with the neighbors. The model becomes analogues to a Monte Carlo realization of the XY ferromagnet at the limit of vanishing velocities. Large-scale numerical simulations showed that the system emerges in a phase-space domain bordered by a critical line along which the fluctuation of the order parameter diverge. They show that high noise and low particle density leads to a no transport phase, where the average velocity was zero, whereas in low voice and high particle density, the swarm is moving in a particular direction. The transition from a stationary state to a mobile state was given the name, kinematic phase change. Similarly, in 1999 [21], Czirók et al. studied a self-propelled model to describe the transitions during the collective motion of organisms in three dimensions. They showed that flocking in the presence of noise and by controlling the control parameters, both disordered and long-range ordered phases can be observed. Another of the author can be found in [22].

Moreover, Couzin et al. in 2002 [19], used a self-organizing model in three-dimensional space to investigate the spatial dynamics in animal groups. They explain the group level behavioral transitions and showed evidence for collective memory in fish schools and bird flocks during transitions of a group from one collective behavior to another. The model also showed influence in group structure as a result of differences amongst group members, and how simple rules can change the spatial positioning within a group in the absence of the latest information such as the position.

Furthermore, D’Orsogna et al. in 2006 [25], modelled self-propelling biological or artificial individuals that are interacting through pairwise attractive and repulsive forces. They were able to express stability and morphology of organization beginning from the shape of two-body interaction. D’Orsogna et al. categorized different solution types via approximate analogy with canonical dissipative systems. Through coherent theory, statistical mechanical analysis of the model was obtained, and parameters of the Morse-type interaction function were used to derive conditions for \textit{H-stable}, that is, a well spaced versus catastrophic increasing density with increasing population solutions.

In addition, Chuang et al. in 2007 [16], derived a 2D continuum analogue to [25]. A class of swarming problems was studied. In here, the particles evolved dynamically via pairwise interaction potentials and a velocity selection mechanism. Various changes of state as function of the self propulsion and interaction potential parameters was beared by the swarming system. A linear stability analysis on stationary solutions for the continuum model was performed and the results were compared to the individual based ones.

Also treating individuals in swarms as self-propelled particles, but using differential equations to govern there motions, several physicists have considered intriguing situations such as changing a collective behavior to another. Erdmann and Ebeling in 2005 [33], considered models of active Brownian agents via a harmonic attractive potential in a two-dimensional system in
the presence of noise. Through numerical simulations, they found that as the noise intensity was increases, this causes a translating swarm of individuals to transition to a rotating swarm with a stationary center of mass. They also investigated the statistical properties of the swarm dynamics in the presence of weak noise. Furthermore, Forgoston and Schwartz in 2008 [35], improved the Erdmann and Ebeling’s 2005 result with the addition of a time delay. Now, the model possessed a transition that depended on the coupling amplitude size. It was shown via the use of mean field equations without noise that the time delay induced transition is associated with a Hopf bifurcation. Juanico in 2009 [56] considered the role of diversity in self-organized pattern formation in an attractive-repulsive swarm system via differential equations as well.

1.5 The Literature on Swarming by Engineers

In the recent years, work has been done from the engineering perspective such as the formation control of multi-robot teams, that is, centralized, decentralized and distributed control rules for systems of agents subjected to various constraints and autonomous air vehicles. Swarming behavior is important in engineering for various reasons. It is used to develop swarms of autonomous agents and the ideas could be used in controlling natural flocks of animals. There are two steps involved in engineering applications of swarming. The first is the generation of the swarm condition. This is the condition which, when satisfied, will guide the generation of swarm agents which is capable of carrying out the desired outcome. This step is normally problem dependent. The second step involves the fabrication of a behavior or set of behaviors that satisfy the given swarm condition. The goal is to create any set of behaviors which fulfills the swarm conditions.

The literature is inundated with work on engineering applications related to swarms. Tanner et al. in 2003 [137, 138] developed control laws on interacting agents that give rise to tight formations and collision avoidance. In the papers, they investigated the stability properties of a system of multiple mobile agents with double integrator dynamics. Firstly, they looked at smooth control laws for the agents via coordination control scheme. The control laws are a combination of attractive/repulsive and alignment forces which ensures collision avoidance and cohesion and a common steering direction. Their control scheme, the topology of the control interconnections was fixed and time invariant. Classical control theory, mechanics and algebraic graph theory was used to prove the results. Flocking behavior was exhibited via the use of local controllers which established a stable, coordinated flocking motion. Network topology was used to show the stability of the group motion. The sequel of this paper had time variance between the individuals in the group. In here, the state of agents were bounded within a certain neighborhood of the control law. This gave rise to a dynamic control interconnection topology and a switching control law ensuring flocking motion.

Reif and Wang in 1999 [108], introduced the concept of very large scale robotic (VLSR) systems via the distributed autonomous control of VLSR systems. They employed artificial force laws, that is, inverse-power force laws and spring laws, incorporating both attraction and repulsion. The force laws can be distinct and to some degree they reflect the ‘social relations’ among robots. Henceforth, they named their method as the social potential fields method.
An individual robot’s motion is administered by the resultant artificial force imposed by other robots and other components of the system such as obstacles. The approach is distributed in that the force calculations and motion control can be done in an asynchronous and distributed manner. Suzuki and Yamashita in 1999 [136], considered a system of multiple mobile robots in which each robot are anonymous. That is, they all execute the same algorithm and they cannot be distinguished by their appearances. Initially they do not have a common x-y coordinate system. It is a distributed system of anonymous mobile processes in which they all execute the same algorithm and they cannot be distinguished in their appearances. They investigated formation problems of geometric patterns in the plane by the robots by converging the robots to a single point and moving the robots to a single point in finite steps. More information on distributed systems can be found in [34, 153].

Moreover, Kazadi in 2005 [59] explored concepts from swarm intelligence and provided mathematical definitions, hence making it more vigorous. The desired state of the global property was examined which generated requirements for local behavior that generated the global property. This particular methodology allowed a local behavior to be tested theoretically before it is tested empirically. In 2007, Sepulchre et al. [113] proposed a design methodology to stabilize parallel motion and circular motion of particles in fixed relative spacing and fixed relative phases respectively moving at a unit speed. Their result focussed on lower-order parametric family of stability which may lead to design of higher level tasks in group formation.

One of the early works on generating distributed swarm models is by Reynolds in 1987 [109] whereby he describes a distributed behavioral model of a flock of birds via computer based animated simulations. His flocking model is directed to the particle system, where the simulated birds were referred to as being the particles. In this model, the birds flock using the local information about its neighbors, that is, each of the birds are independent and navigates according to its local perception of the dynamic environment. Reynolds work showed that behavior have strengths and these are used to help the boid decide what to do. The problem with Reynold’s model is that it is difficult to measure objectively how valid his results are. He compared the behavioral aspects of the simulated flock with the those of natural flocks and used it to refine and improve his model.

Furthermore, swarming problems can also be solved through the concept of artificial potential functions. Work done on swarming using artificial potential functions includes [108, 79, 123, 116]. The governing principle behind the artificial potential field method is to attach attractive fields to the target and repulsive fields to each of the obstacles in the workspace. The whole workspace is then inundated with positive and negative fields, with the direction of motion facilitated via the motion of steepest descent [60, 73]. The artificial potential field method has been used widely and has addressed a spectrum of issues such as parking, posture and point stabilities and path tracking [76, 111, 146, 114, 129] to solve a wide range of problems permeating from robotic applications, that is, robot navigation and control.

The research on autonomous air vehicles, that is, unmanned air vehicles has increased at quite an alarming rate with a wide variety of shapes, sizes, configurations, and characteristics. It is commonly known as a drone without any human pilot. Its flight is either controlled autonomously by computers in the vehicle, or under the remote control of a navigator. Historically, unmanned air vehicles were simply remotely piloted aircraft, but autonomous control is in-
increasingly being employed. The largest use of autonomous air vehicles is within the military applications. It is also being used in some civil applications such as fire fighting, surveillance of pipe works, etc. They use optimization graph theory approach to find the best set of communication channels that will keep the aircraft in desired formation [48]. They also describe reconfiguration strategies in case of faults or loss of aircraft, with a high degree of autonomy. Further work on autonomous air vehicles can be found in [61]. In here, the authors formulated the control problem for a line-of-sight based formation flight configuration of a leader and a follower aircraft.

1.6 Flocking rules

In literature, the flocking models are built within a framework of three basic rules of steering namely:

(1) Separation (Avoidance): Steer to avoid crowding local flock mates (short range repulsion). Basically, it means that each member, or boid, of a flock tries to keep a minimum distance from every other boid in the flock. Basically, each boid considers its distance to other flock mates in its neighborhood and applies a repulsive force in the opposite direction, scaled by the inverse of the distance. While avoiding flock mates, each of the boid tries to not get too close, or avoid, boids in other flocks.

(2) Alignment: Steer towards the average heading of local flock mates. It means that each boid tries to go in the same direction as the rest of the flock. The boids calculate the average velocity of flock mates in neighborhood and steer towards that velocity.

(3) Cohesion (Attraction): Steer to move toward the average position of local flock mates (long range repulsion). It means that each boid tries to get as close as possible to the rest of the flock (huddle together), that is, it keeps boids together as a group. The boids compute the direction to the average position of local flock mates and steer in that direction.

The flocking rules describe how an individual maneuvers based on the positions and velocities of its nearby flock mates [109, 110]. Although the rules governing each member of a flock are seemingly basic, the collective motion is strikingly spectacular. The superposition of these rules results in the flock mates moving in a particular formation, with a common heading whilst ensuring all possible collision and obstacle avoidances [137], that is, basically a life-like behavior emerges from the flocking rules.
1.7 Holonomic Systems and Constraints

Holonomic systems are now being incorporated into practical systems [72]. The constraints applying to an articulated robot can be expressed as equations relating to the configuration space. These equations can be used to eliminate some parameters and reduce the dimension of the configuration space. These are holonomic constraints [72]. A holonomic system is one in which the number of degrees of freedom are equal to the number of coordinates needed to specify the configuration of the system. If a constraint, which restricts a system from translating and rotating freely is written by the following form, then it is called holonomic

\[ f(q,t) = f(q_1, q_2, \cdots, q_n, t) = 0, \]  \hspace{1cm} (1.2)

where \( q \) is the generalized coordinate (configuration variable) of the system and \( t \geq 0 \) represents time. Generalized coordinates are used to locate a system with respect to a reference frame and may include Cartesian or Spherical coordinates, but also include lengths or angles which can be chosen conveniently to describe the system [72]. For example, the robot manipulator consisting of \( n \) revolute joints, a set of joint angles \( \theta_i \), where \( i = 1, \ldots, n \), are the generalized coordinates. Equation (1.2) always reduces the dimension of the system, that is, if there are \( m \) holonomic constraints of \( q_i \), where \( i \in \{1, 2, \cdots, n\} \), the number of independent coordinates are \( n - m \).

In the field of mobile robots, the term holonomic mobile robot is applied to the abstraction called robots, or base, with regard to the rigid bodies which make up the actual mechanism. Thus, any mobile robot with three degrees-of-freedom of motion in the plane has become known as a holonomic mobile robot. Holonomic mobile robots are desirable because they do not have kinematics motion constraints, which make path planning and control much simpler. In a holonomic system, return to the original internal configuration means return to the original system position. Mobile robots with arbitrary planar velocities or ones with translations are holonomic. They are completely integrable[2].

In applications such as mining, logging and automobile construction, an example of a holonomic system is joint actuators, which are used to control a manipulator. Examples of joint actuators are electric motors, hydraulic and pneumatic actuators, to name a few [72]. In the case of the holonomic manipulator with links, it is apparent that the system requires as many actuators as the number of independent configuration variables \( x_i \) and \( y_i \), where \( i \in \{1, 2, \cdots, n\} \), to power and control each joint \( i \). Installing a large number of actuators to power and control a system tends to be cumbersome, and contributes to the increase in the overall bulkiness of the system, thus impairing the system’s input efficiency. Moreover, the installation of such a large number of actuators proves to be an expensive exercise. Consequently, holonomic systems are undesirable in applications where input efficiency and cost reduction are paramount.

Furthermore, the control and the path planning problem of holonomic systems is the same fundamental problem as free-flying systems, that is, systems that are devoid of kinematic or dynamic constraints, and hence do not raise new fundamental issues such that many basic
planning methods remain applicable [72].

1.8 Nonholonomic Systems and Constraints

Ideas from differential geometry, algebra, and other areas of mathematics have played an important role in this process. Now, we are seeing some of these ideas being applied to new problems in motion control, mechanical design, electronic, and indeed in many other areas.

If a constraint cannot be expressed in the form of equation (1.2), it is nonholonomic. As a paradigm, an inequality condition is nonholonomic. The nonholonomic constraints of mechanical systems are expressed by the following equation in most cases:

\[ f(q, \dot{q}, \ddot{q}, t) = 0, \]  

(1.3)

where \( \dot{q} \) and \( \ddot{q} \) are, respectively, the first-order time derivative and the second-order time derivative of the generalized coordinate of the system. The constraint is nonholonomic because it cannot be integrated analytically, that is, it cannot be condensed to the form of equation (1.2). Nonholonomic constraints do not reduce the dimension of the space of configurations attainable by the robot, but reduces the dimension of the space of possible differential motions, that is, the space of the velocity directions at any given configuration [72].

In most cases we deal with the constraint of the following equation:

\[ f(q, \dot{q}) = 0 \]  

(1.4)

Generally, for mechanical systems \( \dot{q} \) represents all higher order derivatives of the position variables and plays a pivotal role in the analysis of the motion of the system without regard to the forces which cause it, and as such the analysis essentially focuses on the study of position, velocity and acceleration of the system. Consequently, equation (1.4) is said to be a kinematic constraint. On the other hand, \( \ddot{q} \) deals with all higher order derivatives of translational and rotational velocity variables, and takes into account the forces that cause the motion of mechanical systems. Accordingly, equation (1.3) represents a dynamic constraint. Control issues of nonholonomic systems under dynamic constraints can be found in [5].

Equation (1.4) can be understood as a velocity constraint of the system at a given configuration. The non-integrable constraint does not automatically reduce the dimension of the configuration space. There arises a possibility to steer the generalized coordinates using less number of inputs, which is unlikely in the holonomic systems [72].

As an example, consider the two-wheeled nonholonomic mobile robot, as depicted in Figure
1.1. Its kinematic model is given as

\[
\begin{align*}
\dot{x} &= v \cos \theta, \\
\dot{y} &= v \sin \theta, \\
\dot{\theta} &= \omega,
\end{align*}
\]

(1.5)

where the \((x, y)\) coordinate gives the location of the center of the axle, \(\theta\) gives the car’s orientation with respect to the \(z_1\) axis of the \(z_1 - z_2\) coordinates while \(v\) and \(\omega\) are, respectively, the translational and rotational velocities of the two-wheeled mobile robot.

At any instant during a motion, assuming no slippage between the two wheels and the surface, the velocity of the two-wheeled mobile robot has to point along the main axis of the robot. That is, the lateral (tangential) velocity, denoted as \(v_L\) in Figure 1.3, of the two wheels must always be zero at any instant during a particular motion.

Eliminating the input \(v\) in equation (1.5) leads to the following nonholonomic constraint,

\[
\dot{x} \sin \theta - \dot{y} \cos \theta = 0.
\]

(1.6)

Equation (1.6) is non-integrable and implies no-side slip condition under the assumption of rolling contact.
With reference to the two-wheeled mobile robot, the nonholonomic constraint represented by equation 1.6 restricts the robot’s motion in directions that are tangential to the wheels at any given time and does not reduce the dimension of the configuration space. Thus, nonholonomic constraints limit only the freedom of motion of a system. Nonholonomic constraints raise three main issues [72]:

(a) First, given a possibly nonholonomic constraint, how can we be convinced that it is actually a nonholonomic one.

(b) Secondly, we need to determine whether the nonholonomic constraints impede the set of configurations reachable from a given configuration.

(c) Last but not least, we need to construct an effective motion planner capable of generating feasible free paths for a robot subject to given nonholonomic constraints.

Nonholonomic systems subject to constraints that are often expressed in terms of non-integrable velocity relationships exhibit remarkably rich behavior. The key attribute of nonholonomic systems is the inclusion of the nonholonomic constraints in the equations of motion of the system. The nonholonomic constraints imposed on the motion of such a system lead to the following two well documented features, different from those of holonomic systems. One is that the equilibrium positions of a nonholonomic system are not isolated, and the set of all the equilibrium positions is a manifold having dimension not less than the number of the nonholonomic constraints. This feature led to two main research focuses: the stability of the manifold of equilibria and the stability of single equilibrium positions in the manifold [156]. Another feature is that the characteristic equations of the first approximation of the equations of motion have zero roots whose number at least equals that of nonholonomic constraints [156].

The two features make the synthesis of control laws and stability of nonholonomic systems very complex, and an active challenging research area as well.

Furthermore, there is a strong presence of mechanical systems that have non-integrable constraints such as robot manipulators, mobile robots, wheeled vehicles, and space and underwater robots. These nonholonomic systems have wide-ranging capabilities and relieve humans of tasks that fall under the four D’s of robotics: dirty, dull, dangerous or difficult [23], and hence as such is another feature that is invariably attracting monumental attention from researchers, especially in the field of robotics.

1.8.1 Work Done on Nonholonomic Systems

The nonholonomic motion planning problem has sparked a series of challenging research activities and researchers, over the years, have produced copious algorithms in conjunction with constructive nonlinear control for tackling this problem. A few of these algorithms are briefly discussed below:

Murray and Sastry [93] proposed a nonlinear control scheme which constructively steers several classes of nonholonomic systems using sinusoids and Fourier analysis. The authors referred to
the classes of steerable systems as "chained form systems". Instead of using single sinusoidal inputs which tend to be time-consuming, sinusoidal inputs which can steer a system all at once were presented. The scheme was applied to a simple hopping robot consisting of a leg which could both rotate and extend. Although the approach of sinusoidal input trajectories is simple, as discussed by the authors, the approach is found wanting in some classes of nonholonomic systems. Consequently, the applicability of the scheme depends on the conversion (if possible) of a system into the aforementioned chained form system.

Gurvits and Li [51] presented an algorithm for computing time-periodic feedback solutions for nonholonomic motion planning with collision avoidance. The basic idea is to use averaging techniques to compute asymptotic behaviors of nonholonomic systems under application of a class of high oscillatory inputs and compute time-periodic feedback solutions for nonholonomic motion planning based on the potential field method. Based on the averaging techniques, the authors provided error bounds between a nonholonomic system and its averaged system together with the accompanying analytical solutions using Fourier analysis.

Sharma and Vanualailai [114] developed an efficient algorithm that considers the multi-task of control and motion planning of a car-like mobile robot, and its posture stabilization within a constrained and obstacle-ridden environment. The authors demonstrated collision-free parking maneuvers considering both kinematic and dynamic constraints via a Lyapunov-based control scheme based on Lyapunov’s Direct Method.

1.9 Motivation

There has been a vast increase of research research done in the field of motion planning and control. This is based greatly on technological advances that is taking place. Motion planning and control problem is directly related to many robotic applications and in general programming complex automatic systems. Today, many manufacturers choose to use robots in their factories to decrease the cost of production. Robots have started taking the place of doctors in several fields of medical surgery namely the ones that require perfect precision such as brain surgeries. However, the applications of robot motion planning and control are not limited with the ones directly related to robotics. Motion planning and control can be used in molecular biology to model the motion of the molecule, especially protein folding which is a current open problem. In computer graphics, motion planning and control is one of the key concepts in creating virtual environments and allowing characters to navigate through such environments. Some special questions of motion planning and control apply to vehicles, aircraft and spacecraft navigation. Although there are many more applications to motion planning and control the ones listed above are some of the most important ones.
The model proposed in this thesis is individual-based. We consider spacing between the individuals, each of which moves with the velocity of the swarm centroid, of primary importance. We create attraction and repulsion functions and use them to move the individuals toward the swarm centroid and for collision-avoidance between the members of the swarm. The swarming behavior is captured by a system of first-order ODEs whose components are derived from the gradient of a Lyapunov-like function that is made up of these attractive and repulsive functions. Therein lies the major contribution of this thesis, and that is, the Lyapunov-like function guarantees the practical stability of a swarm model.

Historically, the concept of practical stability was introduced in 1961 by LaSalle and Lefschetz [71]. They were interested in whether, for a particular system governed by differential equations, the system state would evolve and remain within certain bounded subsets of the state-space, and whether it was possible to estimate bounds of these subsets over a finite or an infinite time. If yes, then an unstable equilibrium point, for example, would not be the most important consideration, but rather the system’s behavior in the vicinity of the equilibrium point would be. The acceptability of the system’s behavior could be measured by the bounds on the subsets of the state-space. Indeed, since practical stability is defined with respect to the bounded subsets rather than an equilibrium point, there need not exist any equilibrium point in the subsets.

Herein lies the strength of practical stability when applied to Lagrangian swarm models for which the centroid is an attractive point. The centroid could be non-stationary or unstable in the sense of Lyapunov if there were unequal attractive and repulsive forces among the individuals in the swarm. In such a situation, it makes more sense to analyze the acceptability of the model’s response in the vicinity of the centroid. In literature, it is only recent that an attempt has been made to study the practical stability of swarm models. A paper by Chen et al. [13] in 2006 and two papers by Pan et al. [101, 102] in 2008 presented models which are practically stable. However, the models are the original Gazi-Passino swarm model which has already been shown to be Lyapunov stable. The authors also considered models which contain several types of bounded perturbation added to the original stable Gazi-Passino swarm model. Even though such perturbed systems with a stable principle component are known to be at least bounded - and hence practical stability criteria could be developed (see, for example, [147]) - the authors did not provide the biological motivations behind the necessity of such perturbed systems.

In this thesis, we utilize a different approach to ensure practical stability of our model, which is not derived from the Gazi-Passino model. We use the results in the monograph by Lakshmikantham et al. [69]. They presented a systematic study of the theory of practical stability via the use of Lyapunov-like functions and comparison first-order ordinary differential equations. They showed that if a comparison equation was practically stable, then the differential system under consideration was also practically stable. Note that the Lyapunov-like functions have properties that differ significantly from the usual Lyapunov functions in that neither the Lyapunov-like functions nor their orbital derivative are required to be sign-definite.

The approach used to constructing a Lyapunov-like function is a result of a development of a Lyapunov-based robot control technique that was proposed in 1990 by Stonier [133], who based his work on an application of the Lyapunov method to qualitative differential games that
In this thesis, a new Lagrangian swarm model is developed by utilizing the hypothesis that swarming is an interplay between a long range attraction and a short range repulsion between the individuals in the swarm [41, 56]. This behavior leads to aggregation and formation, which are important for the survival of the members of the swarm. This attractive-repulsive behavior is modelled by a system of first-order ordinary differential equations (ODEs) that is a gradient of some artificial or social potential function. The aim is to show that this function is a Lyapunov-like function for the system of ODEs and provides the cohesiveness of the swarm in which the distances between the individuals in the swarm are bounded from above, meaning that the members of the swarm tend to stay together and avoid dispersion. The size and the density of the swarm can therefore be measured.

The next section presents an outline and the objectives of the thesis.

1.10 Structure and Objective of this Thesis

In Chapter 2, we present a review of the Lyapunov-based Control Scheme proposed by Sharma in 2008 [114]. Also, we give a brief overview of the Direct Method of Lyapunov and provide definitions of stability and theorems in the Lyapunov sense. Moreover, we discuss the notion of practical stability by Lakshmikantham, Leela and Martynyuk [69]. Then we design and construct our the three-dimensional swarm model by setting up a system of first-order ODEs made up of the instantaneous velocities of the individuals in the swarm. Then, a detailed discussion was done on the properties of the swarm model.

In chapter 3, we construct a Lyapunov-like function that is made up of attraction and repulsion components, and use it to derive the exact form of the instantaneous velocity controllers of each individual. In doing so, we obtain our three-dimensional swarm model, which is a practically stable gradient system. In the same section, we show how the Lyapunov-like function can be used to approximate the size and density of the swarm. The velocity controllers provide a collision-free trajectory within the workspace. Efficiency of the controlled framework and the resulting nonlinear controllers will be verified numerically through computer simulations of interesting emergent swarm behaviors.
In Chapter 4, we attempt to generalize the results of Chapter 3, where we focus on generalizing the swarm model to include obstacles. We construct the Lyapunov-like function that is made up of attraction and repulsion components, and use it to derive the exact form of the instantaneous velocity controllers of each individual. Then we provide the practical stability analysis of the model. The efficiency of the new controllers will be demonstrated via computer simulations of different emergent swarm behaviors.

In Chapter 5, we attempt to generalize further the results of Chapter 4. A new motion planner is designed which provide a feasible solution to the findpath problem and manages simultaneously collision and obstacle avoidances of the multi-vehicle system within a dynamic environment. In this chapter, the dynamic obstacles include a swarm of point masses and car-like robots. For brevity, we will consider only a two-dimensional swarm model for this section of the thesis. New continuous time-invariant acceleration control laws, formulated via the Lyapunov-based control scheme, guarantee the stability of a system of differential equations governing motion planning and control of multiple nonholonomic car-like robots. The effectiveness of the control scheme and the proposed nonlinear controllers, which take into account kinodynamic constraints, are demonstrated via computer simulations.

Chapter 6 gives a brief summary of the landmark results of this thesis, and outlines future work in this area.
Lyapunov-Based Control Scheme and Stability in the Lyapunov Sense

"Each problem that I solved became a rule which served afterwards to solve other problems."
René Descartes (1596 - 1650)

2.1 History

Problems on stability appeared for the first time in mechanics during the investigation of the equilibrium state of systems [87]. The interest in the stability of motion is today greater than ever and hence not only confined to mechanics [77]. Stability plays an important role in economical models, numerical algorithms, quantum mechanics, nuclear physics, nonlinear dynamics and is successfully applied in the field of swarming [77]. It was only in 1644, when E. Torricelli, an Italian physicist and mathematician, formulated the criterion for stability of rigid bodies in equilibrium under gravitational forces. G. Lagrange, in 1788, proved a theorem that defined conditions for stability of the equilibrium point for any conservative system. Lagrange’s Theorem stated that if the system is conservative, a state corresponding to zero kinetic energy and minimum potential energy is a stable equilibrium point. Later in the middle of the 19th Century, problems were encountered in technology in relation to its stability and more general theory of the stability of motion were required rather than specific ones. Criterion for the stability of motion had to be reformulated first before any solutions for problems in technology and mechanics could be achieved.

Lagrange, in 1788, also gave a definition on the stability of an equilibrium which stated that an equilibrium is stable when neighbouring solutions remain close to the equilibrium, which coincides with the concept of stability in the sense of Lyapunov. Moreover, the concept of motion stability was discussed and papers were published from a more general point of view towards the end of the 19th Century by E. J. Routh during 1877 - 1884 and N. E. Zhukovski in 1882. They investigated problems in motion stability using different methods. However, the
major drawback of the researchers of those times was that in the analysis of perturbed motion they only considered linearized equations of perturbed motion and did not consider equations of higher order terms. The results of the analysis of linear equations had only a few things in common with the result of the exact equations. A new set of equations was then needed to be used to tackle this problem and differential equations was then brought into the picture.

Differential equations are essential tools in scientific modelling of physical problems which find relevance in almost every sphere of human endeavor from Agricultural Sciences, Engineering, Medical Science, and Physical sciences to Social sciences. In the development of the subject of differential equations, one may distinguish two broadly distinct streams. On the one hand, there is the endeavor to obtain a definite solution, or one of definite type, either "in closed form," which is rarely impossible, or else by some process of approximation [71]. On the other hand, abandoning all endeavors to reach an exact or approximate solution, one strives to obtain information about the whole class of solutions. This is the qualitative theory, initiated by Poincaré around 1880 and pursued with varying degrees of energy since then [71].

A major problem in qualitative theory is this: given a solution, what is the relation to its neighbors? A solution is a curve or trajectory $C$ in some space. One asks then whether the trajectories starting near a curve do tend to remain near the curve (stable) or to depart from the curve (unstable). This plants squarely the problem of stability within the qualitative theory and here it is not too much to say that the true creator of a stability theory is Lyapunov.

Alexandra Mikhailovich Lyapunov (1857 - 1918) was a Russian mathematician, mechanical engineer and a physicist whose work concentrated on the stability of equilibrium and motion of mechanical system and the stability of a uniformly flowing fluid. He devised important methods of approximation and laid the foundation for stability theory. The two approaches that he proposed are commonly known as Lyapunov’s first method and Lyapunov’s second method. The two approaches will be discussed in detail in the latter section.

The remainder of the chapter is organised as follows: Section 2.2, discusses the stability of dynamical systems; in Section 2.3, we present an overview of the Lyapunov-based Control Scheme proposed in [114], essentially an Artificial Potential Field method; in Section 2.4, we provide the stability definition in the Lyapunov sense; in Section 2.5, we provide a brief introduction to practical stability; in Section 2.6, a two-dimensional swarm model is constructed as an example; and finally, Section 2.7 closes the chapter with an application of the Lyapunov-based Control Scheme.

### 2.2 Stability Theory of Dynamical Systems

The concern with this thesis is dynamical systems. A dynamical system is a collection of objects subjected to some law of force. This systems are governed by a finite number of real parameters and whose performance is described by a finite set of ordinary differential equations. This leads to a finite set of differential equations which governs this motion, which may
not necessarily be linear. Stability theory is fundamental to the study of dynamical systems since the minimum requirements in fields such as mathematics, engineering, non linear mechanics, thermal and robotics is systems that are unperturbed in the presence of unknown disturbances and noises, that is, stable systems [17, 18, 114, 120, 124, 125]. These dynamical systems are used effectively in qualitative examination of their stability and adaptability to the external stimuli or disturbances. The area of mathematics which deals with the behavior of solutions is referred to as stability theory. A general framework for stability analysis of nonlinear, or linear, dynamical systems includes Lyapunov’s First and Second Methods, named after the Russian Mathematician Alexander Mikhailovich Lyapunov [36].

Lyapunov, in 1892, set forth the general framework for the solution of linear and nonlinear systems. His work provided answers to various problems in motion stability and his definitions were so convenient, that it has been universally accepted as the basic definition. It has lead many researchers to find solutions to stability problems that were essentially not solvable at a time. He provided solutions for a large class of steady and unsteady motion as well as periodic motion. Hence, Lyapunov suggested his two most powerful methods for analyzing stability problems of motion, namely the Lyapunov’s First Method and Lyapunov’s Second method. Of these, Lyapunov’s Second Method is widely used due to its efficiency as no other which can provide any good information on nonlinear systems.

2.2.1 The First and Second Methods of Lyapunov

Lyapunov’s first method was developed to analyze systems as a set of differential equations [37]. These method consists of all those methods in which the differential equation of a system can be solved. It involved linearizing the system, including expansion of functions in power series and dropping nonlinear terms. This linearized system was then used to approximate solutions to the original system. Stability or instability was then obtained from this solution.

Lyapunov did point out in his first method that the solution may be obtained in the form of a series from which stability can be determined. The extent of stability, for example, the size of the region of attraction and the behavior of transients as they approach the equilibrium is determined by the nonlinearity of the system [107]. However, in many nonlinear systems, stability results achieved using this method are characteristically localized, and the stability region is difficult to estimate and the region is usually diminutive. Consequently, Lyapunov’s Second Method was developed which takes nonlinearity into account [114].

The second method of Lyapunov is one of the most powerful tools in the analysis of motion stability of nonlinear systems governed by differential equations. It basically concerns the determination of stability information about a nonlinear system, that is the behavior of the system without actually having to solve its differential equation. By manipulating the coefficients of the characteristic equation of the set of ODEs in a prescribed manner, we can determine the polarity of the system, without actually solving the equations directly [129]. Since the method yields stability information directly, it is also commonly known as Lyapunov’s Direct Method. This method utilizes essentially generalized energy like functions called Lyapunov functions. It is with this function that the stability analysis of systems is analyzed.
Though Lyapunov’s method is a powerful way of analyzing the stability of nonlinear systems, the major weakness of Lyapunov’s Second Method is that there is no scheme or coherent methodology available to construct the Lyapunov functions. While the literature on constructing Lyapunov functions is inundated with various methods, commonly used methods utilize first integrals, quadratic forms, variable gradients, matrices, sum of squares decompositions, and trial and error approaches [14, 47, 65, 67, 104, 112, 130]. However, these methods work best in particular situations and mostly have huge restrictions attached.

Skowronski in 1990 [130] showed that in most practical problems, the trial and error procedure works best. This involved guessing a function and checking whether it satisfies the conditions of the Lyapunov method. If not, then another function is considered. One method is to simply take a function, based on energy or other consideration containing some free parameters, calculate its time-derivative and then adjust its parameters until the Lyapunov function and its time-derivative satisfy the required conditions [106, 130].

The stability analysis of most of the control systems in mathematics, science and engineering is analyzed via Lyapunov’s method. Today Lyapunov’s method has gained a worldwide recognition in various fields of studies as a facilitator to exact conditions for stability.

One of the advantage of using the Direct Method of Lyapunov is that it is like killing two birds with one stone. That is, the controllers that govern a particular system is obtained as well as the issue of stability is discussed and solved all in one. The major advantage of Lyapunov’s method is that the stability analysis can be done without any prior knowledge of the solutions of the system [114].

The Lyapunov stability concepts addressed in the following section will be frequently utilized by the LbCS to address the stability issues of the swarm of boids used in this thesis.

In the next section, we present an overview of the Lyapunov Based Control Scheme proposed in [114], essentially an Artificial Potential Field method. This control scheme provides a simple but effective means of constructing continuous control laws for dynamical systems, either linear or nonlinear.

### 2.3 Lyapunov-based Control Scheme

This section reviews and provides an algorithmic representation of the Lyapunov-based control scheme proposed by Sharma in 2008 [114], which will form a theoretical basis of the emergent behavior of the swarm. Operations within the control scheme are guided by the principles of the Direct Method of Lyapunov [114] and hence, the scheme is appropriately classified as a Lyapunov-based control scheme (LbCS), essentially an artificial potential fields method. The main idea behind artificial potential field methods is finding a function that represents the energy of the system, and generating a force so that the energy of the system is minimised and reach it’s minimum value, preferably only, at the goal position.
Table 2.1 provides an algorithmic representation of LbCS, as given in [114]. The steps will be deployed in this research in order to generate a set of feasible control laws for the swarm of boid, with attractive-repulsive inter-individual interaction, assuming that the swarming behavior is a result of the interplay between a long-range attraction and a short-range repulsion between the individuals in the swarm, with the centroid being the center of attraction.

2.4 Stability in the Lyapunov Sense

Adopted from [129].

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with the Euclidean norm $\| \cdot \|$. Let $x = (x_1, x_2, \ldots, x_n)$ denote an element of $\mathbb{R}^n$. Consider an autonomous nonlinear system

$$
\dot{x} = F(x), \quad x(t_0) = x_0, \quad t_0 \geq 0, \quad (2.1)
$$

where $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be smooth enough to guarantee the existence, uniqueness and continuous dependence of solutions $x(t) = x(t; t_0, x_0)$ of (2.1) in $\Omega$, an open and connected set in $\mathbb{R}^n$.

For the purpose of considering stability concept in the sense of Lyapunov, the following assumption is made:

**Assumption 2.4.1.** There is a point $x^* \in \mathbb{R}^n$ such that $F(x^*) \equiv 0$. Then $x(t) \equiv x^*$ is trivially a solution of (2.1) through $x^* \in \Omega$ for all $t \geq t_0$. We call $x^*$ an equilibrium point of system (2.1).

With the above assumption, the following definitions, in the sense of Lyapunov, can be made:

**Definition 2.4.1.** The equilibrium point $x^*$ at time $t_0$ of (2.1) is said to be stable at time $t_0$ if, for each $\epsilon > 0$, there exists a $\delta(t_0, \epsilon) > 0$ such that

$$
\|x(t_0)\| < \delta(t_0, \epsilon) \Rightarrow \|x(t)\| < \epsilon, \forall t > t_0.
$$

It is said to be uniformly stable over $[t_0, \infty)$ if, for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that

$$
\|x(t_1)\| < \delta(\epsilon), t_1 \geq t_0 \Rightarrow \|x(t)\| < \epsilon, \forall t > t_1.
$$

**Definition 2.4.2.** The equilibrium point $x^*$ at time $t_0$ of (2.1) is said to be unstable if it is not stable at time $t_0$.

**Definition 2.4.3.** The equilibrium point $x^*$ at time $t_0$ of (2.1) is said to be asymptotically stable at time $t_0$ if it is stable at time $t_0$, and if there exists a $\delta_1(t_0) > 0$ such that

$$
\|x(t_0)\| < \delta_1(t_0) \Rightarrow \|x(t)\| \to 0 \text{ as } t \to \infty.
$$
Table 2.1: An algorithmic representation of the Lyapunov-based control scheme (LbCS) for motion planning and control of a holonomic swarm of boids and the nonholonomic mobile planar robots. Modified from [114].

<table>
<thead>
<tr>
<th>Steps</th>
<th>Purpose</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Model</td>
<td>The Lagrangian Swarm system are modelled using the generalized ODEs of the form $\dot{x} = F(x)$, $x(t_0) = x_0$, $t_0 \geq 0$.</td>
<td></td>
</tr>
<tr>
<td>2. Attraction Function</td>
<td>A positive function is designed which is intended to be measure of an Euclidian distance from a point mass or a planar robot of a Lagrangian swarm model. This is an agent-based model following the individual agents that make up the swarm. The individuals are attracted towards each other and also form a cohesive group by first having a measurement of the distance from the $i$th individual to the swarm centroid. This function is known as the attraction to the centroid function. To ensure convergence of the swarm to the centroid the usual form of the target attractive function is sufficient to be considered as a suitable attractive potential field function.</td>
<td></td>
</tr>
<tr>
<td>3. Avoidance Function</td>
<td>A positive function is designed which is a Euclidian distance of Lagrangian swarm model to an obstacle. This function is known as the obstacle avoidance function. To ensure avoidance of the obstacle, the Lyapunov based control scheme (LbCS) adopts a strategy inspired by the work done by Tanner in [140] to develop a function that basically is a ratio that encodes the obstacle avoidance function into its denominator. The numerator of the ratio consists of a tuning parameter. This ratio is known as the repulsive potential field function.</td>
<td></td>
</tr>
<tr>
<td>4. Auxiliary Function</td>
<td>To ensure that the Lyapunov function vanishes at the target configuration, the repulsive potential field functions are multiplied by an auxiliary function. The form of this auxiliary function is similar to the attraction to the centroid function designed in Step 2.</td>
<td></td>
</tr>
<tr>
<td>5. Lyapunov Function</td>
<td>In this penultimate step, the Lyapunov based control scheme requires the summation of all the attractive and repulsive potential field functions to form a scalar function, referred to as a Lyapunov function. This function can also be classified as an artificial potential field function or the total potentials. While the attractive potential field functions attach the swarms to the centroid, the repulsive potential field functions attach repulsive fields to the obstacles for successful avoidances.</td>
<td></td>
</tr>
<tr>
<td>6. Control Laws</td>
<td>To derive the continuous control laws, the LbCS differentiates the Lyapunov function, collects the velocity components separately and substitutes the ODEs from Step 1 governing the Lagrangian swarm. With the introduction of convergence parameters, the control laws are developed provided the time derivative of the Lyapunov function is semi-negative, or negative definite.</td>
<td></td>
</tr>
</tbody>
</table>
It is said to be uniformly asymptotically stable over \([t_0, \infty)\) if it is uniformly stable over \([t_0, \infty)\) and if there exists a \(\delta_1 > 0\) such that

\[
||x(t_1)|| < \delta_1, \ t_1 \geq t_0 \Rightarrow ||x(t)|| \to 0 \text{ as } t \to \infty.
\]

Moreover, the convergence is uniform with respect to \(t_1\).

Lyapunov's Direct Method can then be summarized in the following theorem, where \(\mathbb{R}^+ := [0, \infty)\):

**Theorem 2.4.1.** Let \(x^*\) be an equilibrium point of (2.1) and let \(V: \Omega \to \mathbb{R}^+\) be a \(C^1\) function defined on some neighborhood \(\Omega\) of \(x^*\) such that

(i) \(V(x^*) = 0\);

(ii) \(V(x) > 0\) for \(x \in \Omega \setminus \{x^*\}\); and

(iii) \(\dot{V}(x) \leq 0\) for all \(x \in \Omega\). Then \(x^*\) is stable. If \(\dot{V}(x) \leq 0\) is replaced by \(\dot{V}(x) < 0\), then \(x^*\) is asymptotically stable. If \(x^*\) is asymptotically stable, and if, furthermore \(V(x)\) is radially unbounded (that is, \(V(x) \to \infty\) as \(\|x\| \to \infty\)), then \(x^*\) is globally asymptotically stable.

For our application, where we desire the convergence of trajectories to equilibrium points, the Direct Method of Lyapunov tells us that stability ensures only boundedness of solutions in a neighborhood of \(x^*\). However, it is known that if in addition \(F(x) = (f_1(x), \ldots, f_n(x))\) is bounded for \(x\) bounded, then whenever \(d[V(x)]/dt < 0\) for \(x \neq x^*\) and \(d[V(x^*)]/dt = 0\), the equilibrium point \(x^*\) is not only stable, but also attracts trajectories to it. That is, \(x^*\) is asymptotically stable, which is clearly more desirable than stability from a practical point of view.

We refer to \(V(x)\) in Theorem 2.4.1 as a Lyapunov function for system (2.1) and its gradient is the vector field

\[
\nabla V = \left( \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right).
\]

Hence, we will utilize the Lyapunov-based control scheme to design a suitable Lyapunov-like function to carry out the practical stability analysis of a three-dimensional swarm.

### 2.5 Practical Stability

The notion of practical stability was first introduced by LaSalle and Lefschetz in 1961 [71]. It dealt with the question that whether a system state evolves within certain subsets of the state space. Practical stability attracts much attention and have been studied extensively and found many applications in several areas [134, 135, 155]. As pointed out by [70], in some cases, though a system is stable or asymptotically stable in theory, it’s actually unstable in practice.
because the stability domain or the attraction domain is not large enough to allow the desired deviation to cancel out. On the other hand, a desired state of a system may be unstable in the sense of Lyapunov stability, but their performance can be acceptable in practice because the system oscillates sufficiently near this desired state. To deal with these situations, the concept of practical stability is very useful. The notion of practical stability of dynamical systems was first discussed by Lasalle and Lefshetz [71] in 1960s and since then great progress has been made. Practical stability is very useful in estimating the worst-case transient and the steady-state responses and in verifying in time constraints imposed on the state trajectories.

The theory of stability in the sense of Lyapunov is now well known and is widely used in concrete problems in the real world. It is obvious that, in applications, asymptotic stability is more important than stability. In fact, the desirable feature is to know the size of the region of asymptotic stability so that we can judge whether or not a given system is sufficiently stable to function properly and may be able to see how to improve its stability. From a practical point of view, a system is considered stable if the deviations or fluctuations of the motion from the equilibrium remain within certain bounds determined by the physical situations.

On the other hand, the desired system may be mathematically unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. For example an aircraft may oscillate around a mathematically unstable path, yet its performance may be acceptable. Thus, it is clear that we need a notion of stability that is more suitable in several situations than the Lyapunov Stability. Such a concept is called practical stability [69].

Practical stability is concerned with quantitative analysis while the Lyapunov analysis is qualitative in nature. The methodologies proposed to study and verify practical stability are based on Lyapunov-like functions. The Lyapunov-like function is a generalized version of the usual Lyapunov function in the sense that neither the Lyapunov-like function nor its orbital derivative are required to be sign-definite. To study the practical stability of a system, we should first know:

(i) the scope of the permissible state set; and

(ii) to what extent we can control the initial conditions [154].

2.6 A Two-Dimensional Swarm

As an example, we shall construct a two-dimensional model of a swarm with \( n \) individuals moving with the velocity of the swarm's centroid. At time \( t \geq 0 \), let \((x_i(t), y_i(t)), i = 1, \ldots, n,\) be the planar position of the \( i \)th individual, which we shall define as a point mass residing in a disk of radius \( r_i > 0 \),

\[
b_i = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - x_i)^2 + (z_2 - y_i)^2 \leq r_i^2\}.
\]  

(2.2)
Let us define the centroid of the swarm as
\[
(x_c, y_c) = \left( \frac{1}{n} \sum_{k=1}^{n} x_k, \frac{1}{n} \sum_{k=1}^{n} y_k \right).
\]

At time \( t \geq 0 \), let \((v_i(t), w_i(t)) := (x_i'(t), y_i'(t))\) be the instantaneous velocity of the \( i \)th point mass.

Using the above notations, we have thus a system of first-order ODEs for the \( i \)th individual, assuming the initial condition at \( t = t_0 \geq 0 \):
\[
\begin{align*}
    x'_i(t) &= v_i(t) \\
    y'_i(t) &= w_i(t)
\end{align*}
\]
\[
(2.3)
\]

Suppressing \( t \), we let \( x_i = (x_i, y_i) \in \mathbb{R}^2 \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^{2n} \) be our state vectors. Also, let
\[
x_0 = x(t_0) = (x_{10}, y_{10}, \ldots, x_{n0}, y_{n0}).
\]

If \( g_i(x) := (v_i, w_i) \in \mathbb{R}^2 \) and \( G(x) := (g_1(x), \ldots, g_n(x)) \in \mathbb{R}^{2n} \), then our swarm system of \( n \) individuals is
\[
\dot{x} = G(x), \quad x_0 = x(t_0).
\]
\[
(2.4)
\]

An equilibrium point of system (2.4) for which (2.3) is the \( i \)th component will be denoted \( x^*_i = (x^*_1, \ldots, x^*_n) \in \mathbb{R}^{2n} \).

### 2.7 An Application of the Lyapunov Based Control Scheme, LbCS

This section will give a brief overview of the Lyapunov-based Control Scheme (LbCS) and how it can be utilized to generate artificial potential fields. As an example, the motion planning problem of a point-mass operating in a rectangular workspace containing a stationary solid elliptic obstacle shall be taken into consideration. Point masses are the building blocks with which more complex and realistic models of physical systems are constructed. Henceforth, the following definition of a point mass is stipulated:
Definition 2.7.1. The point-mass $P_M$ is a disk of radius $r_P \geq 0$ and is positioned at $(x(t), y(t)) \in \mathbb{R}^2$ at time $t \geq 0$. Precisely, the point-mass is the set
\[ P_M = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - x)^2 + (z_2 - y)^2 \leq r_P^2\}, \]
with respect to the $z_1$-$z_2$ plane.

In this case, the principle objective is to construct a total potential function using the LbCS such that the point-mass $P_M$ steers and reaches a neighborhood of its destination or target configuration in $\mathbb{R}^2$ whilst avoiding a stationary solid object intersecting its path.

### 2.7.1 Target Attraction of the Point Mass

To the point-mass, we will need to assign a target. As such, we define the following definition of the target is stipulated:

Definition 2.7.2. The target of the point-mass is a disk with center $(\tau_1, \tau_2)$ and radius $r_T$. It is described as the set
\[ T = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - \tau_1)^2 + (z_2 - \tau_2)^2 \leq r_T^2\}. \]

Now, in the target-attraction component of the Lyapunov function, intuitively, we want to measure, at time $t \geq 0$, the midpoint position of the point-mass from its destination $(\tau_1, \tau_2)$ and the rate at which it approaches or moves away from $(\tau_1, \tau_2)$. A choice of probable target attractive functions that could accomplish this, on suppressing $t$, is
\[ H_N(x) = \frac{1}{2} \ln(H(x) + 1), \tag{2.5} \]
where
\[ H(x) = (x - \tau_1)^2 + (y - \tau_2)^2 \]
noting that $H_N(x^*) = 0$ and $H_N(x) > 0$ for $x \neq x^*$, $x \in \mathbb{R}^{5 \times n}$. Thus, if the point-mass ever converges to its destination, then it remains there for all time $t$ since $H_N(x^*) = 0$.

The use of $H_N$, together with obstacle-avoidance functions, should enable us to construct Lyapunov function that ensure that system trajectories start and remain close to $x^*$. This form of the target attractive function ($H_N$) will ensure faster convergence to the goal configuration [114].

### 2.7.2 Obstacle Avoidance

Let us fix two stationary elliptic obstacles within the boundaries of the workspace of the point-mass $P_M$. The following definition of the stationary elliptic obstacle is formulated as follows:
Definition 2.7.3. The elliptic obstacle is an ellipse with center \((o_1, o_2)\). Precisely, elliptic obstacle is the set

\[ E = \{(z_1, z_2) \in \mathbb{R}^2 : \frac{(z_1 - o_1)^2}{a_i^2} + \frac{(z_2 - o_2)^2}{b_i^2} \leq 1\}, \]

We note that a horizontal ellipse is when \(a > b\), a vertical ellipse is when \(a < b\), while if \(a = b\) the ellipse collapses into a circle with radius \(a = b\). For the point-mass to avoid the elliptic obstacles we introduce tuning parameter \(\gamma_i > 0\) and consider repulsive potential fields defined by \(U_{rep} : \mathbb{R}^2 \to \mathbb{R}^+\) with

\[ U_{rep}(x) = \sum_{i=1}^{2} \frac{\gamma_i}{FO_i(x)}, \tag{2.6} \]

where the associated obstacle avoidance function is of the form

\[ FO_i(x) = \frac{1}{2} \sum_{i=1}^{2} \left[ \frac{(x - o_1)^2}{(a_i + r_T)^2} + \frac{(y - o_2)^2}{(b_i + r_T)^2} - 1 \right]. \tag{2.7} \]

The tuning parameter \(\gamma_i > 0\) is basically a restriction in the requirement that the desired trajectory is obtained. The restriction is that it be sufficiently be small enough so that the existence of a stable equilibrium state of the system in the neighborhood of the center of a target could be guaranteed. In other words, with this restriction, we get the best possible result at the end of a trajectory, and that is, an object ceases motion very close to the center of its target [146].

2.7.3 The Lyapunov Function as Total Potentials

Let the constant \(\gamma_i > 0\) be a tuning (or control) parameter, and consider as a tentative Lyapunov function that guarantees target convergence and obstacle avoidance as the sum of
the attractive and repulsive potential fields. This is basically a total potential function

\[ L(x) = H_N(x, y) + \sum_{i=1}^{2} \gamma_i F O_i(x). \]  

This Lyapunov function can be easily extended to encompass multi point-mass systems, and it requires the control parameters only for the purpose of controlling the direction of the trajectory.

Consider the situation where the obstacle has center \((a_{11}, a_{12}) = (10, 20)\) with \(a_1 = 2\) and \(b_1 = 1\), and \((a_{21}, a_{22}) = (30, 20)\) with \(a_2 = 2\) and \(b_2 = 2\), and the target has the center \((\tau_1, \tau_2) = (20, 20)\). Let the point mass have radius \(r_t = r_P = 1\) and the control parameters be \(\gamma_1 = \gamma_2 = 0.5\). As an illustration, we show the total potential field generated by equation (2.8) to guide the point-mass \(P_M\) to the target. The sum of two potentials is a quadratic well with its minimum at \((20, 20)\).

![3D Visualization](image1.png)  ![Contour Plot](image2.png)

(a) 3D Visualization  (b) Contour Plot

Figure 2.2: The total potentials and the corresponding contour plot generated from the attractive potentials for the target attraction and the repulsive potentials for the avoidance of the two elliptic obstacles, generated by equation (2.8).

This concludes the section on the Lyapunov-based control scheme and the practical stability of nonlinear dynamical systems in the Lyapunov sense.
Chapter 3

Practical Stability Analysis via a Lyapunov-like Function with Attraction and Repulsion Components

"At bottom, robotics is all about us. It is the discipline of emulating our lives, of wondering how we work." 
Rod Grupen, Discover Magazine, June 2008

3.1 Introduction

This chapter considers the navigation problem of a three-dimensional swarm via an artificial potential fields (APFs) method using the Lyapunov based control scheme (LbCS). It will be shown that the LbCS is equally effective in designing the continuous time-invariant velocity control laws. In essence, we design a motion planner derived from the LbCS, that guarantees the establishment and maintenance of a geometrical formation of a swarm of boids, considering all practical limitations and constraints. This chapter, will in general showcase and mimic the patterns arising from the emergent behavior of the swarms into various forms of simulations. We will present a detailed derivation of the equations of motion of the three-dimensional swarm. Then, by the use of these equations and utilizing the LbCS, we will extract the nonlinear controllers that will result in the emerging swarm behavior.

The remainder of this chapter is organized as follows: in Section 3.2, a three-dimensional swarm model is constructed; in Section 3.3, the attractive and repulsive potential field functions are designed in accordance with the LbCS; in Section 3.4, the velocity controllers are designed and stability analysis of the system carried out; in Section 3.6 we discuss the form of the Lyapunov-like function and its cohesiveness; in Section 3.7 we show how the Lyapunov-like function can be used to approximate the size and density of the swarm; in Section 3.8, we illustrate the feasibility of the proposed controllers via computer simulations of two different scenarios; and finally, Section 3.9 closes the chapter with a brief conclusion.
We will use the following two terms from [90] as we develop our Lyapunov-like function for system (3.3):

1. A *cohesive* group is a group in which the distances between individuals are bounded from above (members of a cohesive group tend to stay together and avoid dispersing).

2. A *well-spaced* group is a group which does not collapse into a tight cluster, i.e., where some minimal bin size exists such that each bin contains at most one individual. Moreover, the size of such a bin is independent of the number of individuals in a group.

### 3.2 A Three-Dimensional Swarm Model and its Practical Stability

We shall construct a model of a swarm with $n$ individuals moving with the velocity of the swarm’s centroid. Following previous work such as those of [90] and [41], we consider the individuals as point masses. For clarity of exposition, we confine ourselves to constructing the three-dimensional version of the model.

At time $t \geq 0$, let $(x_i(t), y_i(t), z_i(t))$, $i = 1, 2, \ldots, n$, be the planar position of the $i$th individual, which we shall define as a point mass residing in a disk of radius $r_i > 0$,

$$b_i = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : (z_1 - x_i)^2 + (z_2 - y_i)^2 + (z_3 - z_i)^2 \leq r_i^2\}. \quad (3.1)$$

The sphere is described in [90] as a *bin*, and in [41] as a *private or safety area* of each individual. We shall use the former term, with *bin size* being the radius $r_i$ of the sphere.

Let us define the *centroid of the swarm* as

$$(x_c, y_c, z_c) = \left( \frac{1}{n} \sum_{k=1}^{n} x_k, \frac{1}{n} \sum_{k=1}^{n} y_k, \frac{1}{n} \sum_{k=1}^{n} z_k \right).$$

At time $t \geq 0$, let $(v_i(t), w_i(t), u_i(t)) := (x'_i(t), y'_i(t), z'_i(t))$ be the instantaneous velocity of the $i$th point mass.

Using the above notations, we have thus a system of first-order ODEs for the $i$th individual,
assuming the initial condition at \( t = t_0 \geq 0 \):

\[
\begin{aligned}
x_i'(t) &= v_i(t) \\
y_i'(t) &= w_i(t) \\
z_i'(t) &= u_i(t)
\end{aligned}
\]  \hspace{1cm} (3.2)

\[x_{i0} := x_i(t_0), y_{i0} := y_i(t_0), z_{i0} := z_i(t_0).\]

Suppressing \( t \), we let \( x_i = (x_i, y_i, z_i) \in \mathbb{R}^3 \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^{3n} \) be our state vectors. Also, let

\[x_0 = x(t_0) = (x_{10}, y_{10}, z_{10}, \ldots, x_{n0}, y_{n0}, z_{n0}).\]

If \( g_i(x) := (v_i, w_i, u_i) \in \mathbb{R}^3 \) and \( G(x) := (g_1(x), \ldots, g_n(x)) \in \mathbb{R}^{3n} \), then our swarm system of \( n \) individuals is

\[
\dot{x} = G(x), \quad x_0 = x(t_0).
\]  \hspace{1cm} (3.3)

Let

\[
x_i^* = (x_c, y_c, z_c) = \left( \frac{1}{n} \sum_{k=1}^{n} x_k, \frac{1}{n} \sum_{k=1}^{n} y_k, \frac{1}{n} \sum_{k=1}^{n} z_k \right), \quad i = 1, 2, \ldots, n,
\]

and

\[
x^* = (x_1^*, \ldots, x_n^*) = \left( \frac{1}{n} \sum_{k=1}^{n} x_k, \frac{1}{n} \sum_{k=1}^{n} y_k, \frac{1}{n} \sum_{k=1}^{n} z_k, \ldots, \frac{1}{n} \sum_{k=1}^{n} x_k, \frac{1}{n} \sum_{k=1}^{n} y_k, \frac{1}{n} \sum_{k=1}^{n} z_k \right).\]

Then we have the Euclidean norm

\[
\|x - x^*\| = \left[ \left( x_1 - \frac{1}{n} \sum_{k=1}^{n} x_k \right)^2 + \left( y_1 - \frac{1}{n} \sum_{k=1}^{n} y_k \right)^2 + \left( z_1 - \frac{1}{n} \sum_{k=1}^{n} z_k \right)^2 + \cdots \right. \\
+ \left. \left( x_n - \frac{1}{n} \sum_{k=1}^{n} x_k \right)^2 + \left( y_n - \frac{1}{n} \sum_{k=1}^{n} y_k \right)^2 + \left( z_n - \frac{1}{n} \sum_{k=1}^{n} z_k \right)^2 \right]^{1/2}.
\]

If \( G \in C[\mathbb{R}^{3n}, \mathbb{R}^{3n}] \), then we can invoke the definition of the practical stability of system (3.3) as provided by [69], noting that we do not need the existence of an equilibrium point of the system. In the definition, \( \mathbb{R}_+ := [0, \infty) \).

**Definition 3.2.1.** System (3.3) is said to be...
(S1) practically stable if given \((\lambda, A)\) with \(0 < \lambda < A\), we have \(||x_0 - x^*|| < \lambda\) implies that \(||x(t) - x^*|| < A, t \geq t_0\) for some \(t_0 \in \mathbb{R}_+\);

(S2) uniformly practically stable if (S1) holds for every \(t_0 \in \mathbb{R}_+\).

The following comparison principle is adapted from [69] to analyse the practical stability of system (3.3),

\[
K = \{ a \in C[\mathbb{R}_+, \mathbb{R}_+] : a(d) \text{ is strictly increasing in } d \text{ and } a(d) \to \infty \text{ as } d \to \infty \},
\]

\[
S(\rho) = \{ x \in \mathbb{R}^{3n} : ||x - x^*|| < \rho \},
\]

and, for any Lyapunov-like function \(V \in C[\mathbb{R}_+ \times \mathbb{R}^{3n}, \mathbb{R}_+]\),

\[
D^+ V(t, x) := \limsup_{h \to 0^+} \frac{V(t + h, x + hG(x)) - V(t, x)}{h},
\]

for \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^{3n}\), noting that if \(V \in C^1[\mathbb{R}_+ \times \mathbb{R}^{3n}, \mathbb{R}_+]\), then \(D^+ V(t, x) = V'(t, x)\), where

\[
V'(t, x) = V_t(t, x) + V_x(t, x)G(x).
\]

**Theorem 3.2.1** (Lakshmikantham, Leela and Martynyuk [69]). Assume that

1. \(\lambda\) and \(A\) are given such that \(0 < \lambda < A\);
2. \(V \in C[\mathbb{R}_+ \times \mathbb{R}^{3n}, \mathbb{R}_+]\) and \(V(t, x)\) is locally Lipschitzian in \(x\);
3. for \((t, x) \in \mathbb{R}_+ \times S(A), b_1(||x - x^*||) \leq V(t, x) \leq b_2(||x - x^*||), b_1, b_2 \in K\) and \(D^+ V(t, x) \leq q(t, V(t, x)), q \in C[\mathbb{R}_+^3, \mathbb{R}_+];\)
4. \(b_2(\lambda) < b_1(A)\) holds.

Then the practical stability properties of the scalar differential equation

\[
h'(t) = q(t, h), \quad h(t_0) = h_0 \geq 0,
\]

imply the corresponding practical stability properties of system (3.3).

### 3.3 Deployment of Lyapunov-based Control Scheme

The principal objective is to construct artificial potential functions (APFs) from a recently developed Lyapunov-based control scheme (LbCS) found in [114] and utilise the procedure outlined in Table 2.1. Accordingly, we derive the velocity controls, \(v_i, w_i,\) and \(u_i\), such that the swarm of boids will be able to exhibit unique swarming behavior in some certain direction. The swarm of boids navigate in the environment whilst respecting the kinematic constraints; and exhibit a swarming behavior. As described in Table 2.1, the APFs proposed are basically
distance functions formed in Euclidean space, a strategy frequently cited in literature. The control scheme appropriately combines these APFs to form a Lyapunov-like function candidate—a platform to design the nonlinear velocity controllers for the swarm of boids. This Lyapunov-like function candidate will also be utilized in a later section to prove stability of the system.

A dichotomy of APFs will be designed in the following subsections: the attractive APFs for convergence and the repulsive APFs that repel the swarm from specified obstacles in a defined workspace.

### 3.3.1 Attraction to the Centroid

We can ensure that the individuals of the swarm are attracted towards each other and also form a cohesive group by having a measurement of the distance from the $i$th individual to the swarm centroid. This is the concept behind *flock centering*, which is one of the well-known three heuristic flocking rules of Reynolds’ [109]. The rule stipulates that the individuals stay close to the nearest flock mates. Basically, the rule minimizes the exposure of a member of a flock to the flock’s exterior by having the member move towards the perceived center of the flock. It is therefore a form of attraction between individuals. Centering necessitates a measurement of the distance from the $i$th individual to the swarm centroid. Thus, we define the following function:

$$R_i(x) := \frac{1}{2} \left[ \left( x_i - \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 + \left( y_i - \frac{1}{n} \sum_{i=1}^{n} y_i \right)^2 + \left( z_i - \frac{1}{n} \sum_{i=1}^{n} z_i \right)^2 \right], \quad i \in \mathbb{N}. \quad (3.4)$$

This will be part of a Lyapunov-like function for system (3.3), and as we shall see later, its role is to ensure that $i$th individual is attracted to the swarm centroid.

### 3.3.2 Inter-individual Collision Avoidance

The short range repulsion between individuals necessitates first a measurement of the distance between the $i$th and the $j$th individuals, $j \neq i$, $i, j \in \mathbb{N}$. With equation (3.1) of the $i$th individual in mind and for the boids to avoid each other, we consider the following function:

$$Q_{ij}(x) := \frac{1}{2} \left[ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (r_i + r_j)^2 \right]. \quad (3.5)$$

The function is an Euclidean measure of the distance between the individual boids, and will appear in the denominator of an appropriate term in the candidate Lyapunov-like function to be proposed.

### 3.4 Design of the Velocity Controllers

The nonlinear control laws for system (3.2) will be designed using the LbCS. In parallel, the control scheme will then utilize Theorem 3.2.1 to provide the mathematical proof of the
practical stability of the system (3.2).

### 3.4.1 Lyapunov-like Function

As per the LbCS, we combine the attractive and the repulsive potential functions. We introduce tuning parameters (or control parameters), that is, let there be real numbers $\gamma_i > 0$, $\beta_{ij} > 0$, and define, for $i, j = 1, \ldots, n$, a Lyapunov-like function for system (3.2) as

$$L_i(x) = \gamma_i R_i(x) + \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}(x)}.$$  \hfill (3.6)

Next, we consider a tentative Lyapunov-like function for system (3.3) as

$$L(x) := \sum_{i=1}^{n} L_i(x_i).$$

It is clear that $L$ is continuous and locally positive definite over the domain

$$D(L) := \left\{ x \in \mathbb{R}^{3n} : \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} Q_{ij}(x) > 0 \right\}.$$

Note that $L(x^*) = 0$. However, $x^* \notin D(L)$ since

$$\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} Q_{ij}(x^*) = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} (r_i + r_j)^2 < 0.$$

This is indeed a desirable situation since if $x^* \in D(L)$, and if at some time $t \geq 0$, we have that $x = x^*$, then this implies that the swarm has collapsed onto itself, a biologically impossible situation. As such, we are not interested in the centroid, but in the behavior of our swarm in the vicinity of its centroid.

### 3.4.2 Nonlinear Velocity Controllers for the Swarm

The time-derivative of $L$ along every solution of system (3.3) is the dot product of the gradient of $L$, given by,

$$\nabla L = \left( \frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial y_1}, \frac{\partial L}{\partial z_1}, \ldots, \frac{\partial L}{\partial x_n}, \frac{\partial L}{\partial y_n}, \frac{\partial L}{\partial z_n} \right).$$
and the time-derivative of the state vector $\mathbf{x} = (x_1, y_1, z_1 \ldots, x_n, y_n, z_n)$. That is,

$$
\dot{L}(\mathbf{x}) = \nabla L(\mathbf{x}) \cdot \dot{\mathbf{x}} = \sum_{i=1}^{n} \left( \gamma_i \dot{R}_i(\mathbf{x}) + \sum_{j=1, j \neq i}^{n} \beta_{ij} \dot{R}_i(\mathbf{x}) - \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{Q_{ij}(\mathbf{x})} Q_{ij}(\mathbf{x}) \right),
$$

where

$$
\sum_{i=1}^{n} \dot{R}_i(\mathbf{x}) = \sum_{i=1}^{n} \left[ \left( x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - \frac{1}{n} \sum_{m=1}^{n} \left( x_m - \frac{1}{n} \sum_{k=1}^{n} x_k \right) \right] x'_i \\
+ \sum_{i=1}^{n} \left[ \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - \frac{1}{n} \sum_{m=1}^{n} \left( y_m - \frac{1}{n} \sum_{k=1}^{n} y_k \right) \right] y'_i \\
+ \sum_{i=1}^{n} \left[ \left( z_i - \frac{1}{n} \sum_{k=1}^{n} z_k \right) - \frac{1}{n} \sum_{m=1}^{n} \left( z_m - \frac{1}{n} \sum_{k=1}^{n} z_k \right) \right] z'_i,
$$

and

$$
\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \dot{Q}_{ij}(\mathbf{x}) = 2 \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} (x_i - x_j)x'_i + 2 \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} (y_i - y_j)y'_i + 2 \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} (z_i - z_j)z'_i.
$$

Noting that $\frac{1}{n} \sum_{m=1}^{n} \left( h_m - \frac{1}{n} \sum_{k=1}^{n} h_k \right) = 0$ for any $h_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, we simplify the former expression to

$$
\sum_{i=1}^{n} \dot{R}_i(\mathbf{x}) = \sum_{i=1}^{n} \left[ \left( x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) x'_i + \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) y'_i + \left( z_i - \frac{1}{n} \sum_{k=1}^{n} z_k \right) z'_i \right].
$$

Now, collecting terms with $x'_i, y'_i$ and $z'_i$, and substituting $x'_i = \dot{x}_i = v_i, y'_i = \dot{y}_i = w_i$
and $z'_i = z_i = u_i$ from system (3.2), we have

$$
\dot{L}(x) = \sum_{i=1}^{n} \left\{ \left( \gamma_i + \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - 2 \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (x_i - x_j) \right\} \dot{x}_i \\
+ \sum_{i=1}^{n} \left\{ \left( \gamma_i + \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - 2 \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (y_i - y_j) \right\} \dot{y}_i \\
+ \sum_{i=1}^{n} \left\{ \left( \gamma_i + \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( z_i - \frac{1}{n} \sum_{k=1}^{n} z_k \right) - 2 \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (z_i - z_j) \right\} \dot{z}_i
$$

$$
= \sum_{i=1}^{n} \left[ \frac{\partial L}{\partial x_i} \cdot \dot{x}_i + \frac{\partial L}{\partial y_i} \cdot \dot{y}_i + \frac{\partial L}{\partial z_i} \cdot \dot{z}_i \right] \\
= \sum_{i=1}^{n} \left[ \frac{\partial L}{\partial x_i} \cdot v_i + \frac{\partial L}{\partial y_i} \cdot w_i + \frac{\partial L}{\partial z_i} \cdot u_i \right].
$$

Let there be real numbers $\mu_i > 0$, $\nu_i > 0$ and $\eta_i > 0$ such that

$$
v_i = -\mu_i \frac{\partial L}{\partial x_i}, \quad w_i = -\nu_i \frac{\partial L}{\partial y_i} \quad \text{and} \quad u_i = -\eta_i \frac{\partial L}{\partial z_i}.
$$

Then

$$
\dot{L}(x) = -\sum_{i=1}^{n} \left[ \mu_i \left( \frac{\partial L}{\partial x_i} \right)^2 + \nu_i \left( \frac{\partial L}{\partial y_i} \right)^2 + \eta_i \left( \frac{\partial L}{\partial z_i} \right)^2 \right] \\
= -\sum_{i=1}^{n} \left[ \frac{v_i^2}{\mu_i} + \frac{w_i^2}{\nu_i} + \frac{u_i^2}{\eta_i} \right] \leq 0,
$$

for all $x \in D(L)$.

For the $i$th individual, system (3.2) therefore becomes

$$
\begin{align*}
x'_i(t) &= v_i(t) = v_i(x(t)) = -\mu_i \frac{\partial L}{\partial x_i}, \\
y'_i(t) &= w_i(t) = w_i(x(t)) = -\nu_i \frac{\partial L}{\partial y_i}, \\
z'_i(t) &= u_i(t) = u_i(x(t)) = -\eta_i \frac{\partial L}{\partial z_i}, \quad \text{for all } \ x_i(t_0), y_i(t_0), z_i(t_0), \ t_0 \geq 0,
\end{align*}
$$

\(3.7\)
where

\[
\frac{\partial L}{\partial x_i} = \left( \gamma_i + \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - 2 \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (x_i - x_j),
\]

\[
\frac{\partial L}{\partial y_i} = \left( \gamma_i + \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - 2 \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (y_i - y_j),
\]

and

\[
\frac{\partial L}{\partial z_i} = \left( \gamma_i + \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( z_i - \frac{1}{n} \sum_{k=1}^{n} z_k \right) - 2 \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (z_i - z_j).
\]

Define the \( n \times n \) diagonal matrix

\[
H = \text{diag}(\mu_1, \nu_1, \eta_1, \ldots, \mu_n, \nu_n, \eta_n),
\]

3n elements

Then system (3.3) becomes the gradient system

\[
\dot{x} = G(x) = -H (\nabla L(x)), \quad x_0 := x(t_0), \quad t_0 \geq 0,
\]

the \( i \)th term of which is given by (3.7). It is clear that \( G \in C[D(L), \mathbb{R}^{32n}] \).

### 3.5 Practical Stability Analysis

In this section, we shall prove the practical stability of system (3.8). Using the method by Lakshmikantham, Leela and Martynyuk [69].

**Theorem 3.5.1.** System (3.8) is uniformly practically stable.

**Proof.** Since

\[
\dot{L}(x(t)) \leq 0,
\]

we have

\[
0 \leq L(x(t)) \leq L(x(t_0)) \quad \forall \ t \geq t_0 \geq 0.
\]

Accordingly, for comparative analysis, it is sufficient to consider the practical stability of the
scalar differential equation

\[ h'(t) = 0, \ h(t_0) = h_0, \ t_0 \geq 0. \tag{3.10} \]

The solution is

\[ h(t; t_0, h_0) = h_0, \]

so that relative to every point \( h^* \in \mathbb{R} \), we have

\[ h(t; t_0, h_0 - h^*) = h_0 - h^*, \]

so that for any given number \( P_0 > 0 \),

\[ |h(t; t_0, h_0 - h^*)| \leq |h_0 - h^*| + P_0. \]

We shall next show that by applying Theorem 3.2.1, we can simultaneously derive the explicit form of \( P_0 > 0 \), with which it is easy to see that (S2) holds for equation (3.10) if

\[ A = A(\lambda) := \lambda + P_0. \]

To apply Theorem 3.2.1, we restrict our domain to \( D(L) \) over which we see that \( L \in C[D(L), \mathbb{R}_+] \), and note that \( L \) is locally Lipschitzian in \( D(L) \) since \( dL/dt \leq 0 \) in \( D(L) \).

Re-defining \( S(\rho) \) as \( S(\rho) = \{ x \in D(L) : \| x - x^* \| < \rho \} \), we get

\[ S(A) = \{ x \in D(L) : \| x - x^* \| < \lambda + P_0 \}. \]

Recalling that \( \gamma_i > 0, \ i \in \mathbb{N} \), we let

\[ \gamma_{\min} := \min_{i \in \mathbb{N}} \gamma_i \quad \text{and} \quad \gamma_{\max} := \max_{i \in \mathbb{N}} \gamma_i, \]

Further, let

\[ b_1(\| x - x^* \|) := \frac{1}{2} \gamma_{\min} \| x - x^* \|^2 \quad \text{and} \quad b_2(\| x - x^* \|) := \frac{1}{2} \gamma_{\max} \left[ \| x - x^* \|^2 + L(x_0) \right]^2, \]

noting that \( b_1, b_2 \in K \). Then assuming \( P_0 > 0 \) we easily see that with (3.9) we have

\[ b_1(\| x - x^* \|) \leq L(x) \leq b_2(\| x - x^* \|) \quad \text{for} \quad x \in S(A), \]

since

\[
\sum_{i=1}^{n} R_i(x) = \frac{1}{2} \sum_{i=1}^{n} \left[ \left( x_i - \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 + \left( y_i - \frac{1}{n} \sum_{i=1}^{n} y_i \right)^2 + \left( z_i - \frac{1}{n} \sum_{i=1}^{n} z_i \right)^2 \right] = \frac{1}{2} \| x - x^* \|^2.
\]

Indeed, the inequality \( b_2(\lambda) < b_1(A) \) yields

\[ \frac{1}{2} \gamma_{\max} [\lambda + L(x_0)]^2 < \frac{1}{2} \gamma_{\min} [\lambda + P_0]^2, \]
which holds if we choose
\[ P_0 > \left[ \left( \sqrt{\frac{\gamma_{\max}}{\gamma_{\min}}} - 1 \right) + \sqrt{\frac{\gamma_{\max}}{\gamma_{\min}}} L(x_0) \right]. \]

Since \( \frac{\gamma_{\max}}{\gamma_{\min}} \geq 1 \) for any \( \gamma_{\max}, \gamma_{\min} > 0 \), and because of (3.9), it is clear that \( P_0 \) exists and \( P_0 > 0 \). Thus, with \( q(t, z) \equiv 0 \), we conclude the proof of Theorem 3.5.1.

3.6 Insight into the form of the Lyapunov-like Function and Cohesiveness

Let us now discuss the idea behind the construction of our Lyapunov-like function
\[ L(x) = \sum_{i=1}^{n} L_i(x) = \sum_{i=1}^{n} \left( \gamma_i R_i(x) + \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}(x)} \right). \]

At large distances between the \( i \)th and the \( j \)th individuals, the ratio,
\[ \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}(x)}, \quad (3.11) \]

is negligible, and the term \( \sum_{i=1}^{n} \gamma_i R_i(x) \) dominates. Then, since \( L(x) \to 0 \) as \( x \to x^* \), the long-range attraction requirement in a swarm model is met, and \( \sum_{i=1}^{n} \gamma_i R_i \) acts as the attraction function; each individual is attracted to the centroid, and therefore the swarm system (3.8) maintains centering and hence cohesiveness at all times. In fact, Theorem 3.5.1 proves the cohesiveness of the swarm, with the boundedness of solution for all time \( t \geq t_0 \) implying that distances between individuals are bounded from above at all times. Note that the parameter \( \gamma_i > 0 \) can be considered as a measurement of the strength of attraction between an individual \( i \) and the swarm centroid, and hence between each other. The smaller the parameter is, the weaker the cohesion of the swarm is. Hence, \( \gamma_i \) can be considered a cohesion parameter.

Consider now the situation where any two individuals \( i \) and \( j \) approach each other. In this case, \( Q_{ij} \) decreases and ratio (3.11) increases, with \( \beta_{ij} > 0 \) acting as a coupling parameter that is a measurement of the strength of interaction between the individuals. In this way, ratio (3.11) acts as an inter-individual collision-avoidance function, because it can be allowed to increase in value (corresponding to avoidance) as individuals approach each other. However, this increase cannot be unbounded in \( D(L) \) because Theorem 3.5.1 shows that every solution \( x(t) \) of system (3.8) is bounded. In other words, collision-avoidance occurs without the danger
of the individuals getting too close to each other, or the swarm collapsing on itself; simply $Q_{ij} = 0$ is not possible in $D(L)$. We have therefore met the short-range repulsion requirement in an individual-based model. Note that the increase in the ratio does not translate to an increase in $L \equiv L(t)$, simply because $L$ is non-increasing in $t$ and any increase in the ratio gives a smaller or the same value of $L$ at time $t$ compared to all previous values of $L$.

The choices of $\beta_{ij} > 0$ determine whether our system is isotropic or anisotropic. If $\beta_{ij} > 0$ are the same for all individuals, then we have an isotropic swarm model. If they differ between at least two individuals, then the model is anisotropic.

Finally, note that we have used three other parameters, $\mu_i > 0$, $\nu_i > 0$ and $\eta_i > 0$ in system (3.8). Because the parameters are a measure of the rate of decrease of $L \equiv L(t)$ at time $t \geq 0$, we name them convergence parameters. The larger the convergence parameters, the quicker the movements of the individuals toward and about the centroid.

### 3.7 Size and Density of the Swarm

Given that a member $i$ of the swarm resides in a sphere defined in (3.1), with radius $r_i > 0$, we can follow the argument by Gazi and Passino [40] to estimate the size and density of the swarm in a stable arrangement, but without using their assumption that the swarm members had to be squeezed cohesively as closely as possible in an area (a disk) of radius $r$, since Theorem 3.5.1 already provides this cohesiveness. Indeed, since Theorem 3.5.1 establishes the practical stability of system (3.8) in $D(L)$, there is no collision between members in $D(L)$. Accordingly, between two members $i$ and $j$

$$\|x_i(t) - x_j(t)\| > (r_i + r_j), \quad x_i = (x_i, y_i),$$

for all time $t \geq t_0 \geq 0$. Now, the safety areas are disjoint, so the total area occupied by the swarm is $\pi \sum_{i=1}^{n} r_i^2$. By Theorem 3.5.1, given $(\lambda, A)$, with $0 < \lambda < A$, we have $\|x(t_0) - x^*\| < \lambda$ implies $\|x(t) - x^*\| < A$ for all $t \geq t_0 \geq 0$. In such a practical stability arrangement, where all the solutions of (3.8) are bounded above by $A > 0$, we can therefore find a disk of radius, say, $p = p(A)$, around the centroid $(x_c, y_c, z_c) = \left( \frac{1}{n} \sum_{k=1}^{n} x_k, \frac{1}{n} \sum_{k=1}^{n} y_k, \frac{1}{n} \sum_{k=1}^{n} z_k \right)$ such that

$$\pi p^2(A) \geq \pi \sum_{i=1}^{n} r_i^2.$$

From this we get

$$p_{\text{min}} := \sqrt{\sum_{i=1}^{n} r_i^2},$$
a lower bound on the radius of the smallest circle which can enclose all the individuals. It is clear that the swarm size will scale with the size of the individual.

If we define the density of the swarm as the number of individuals per unit area, and let it be \( \rho \), then it is simple to see that \( \rho \) is upper bounded, with

\[
\rho \leq \frac{n}{\pi \sum_{i=1}^{n} r_i^2}.
\]

Hence, the swarm cannot become arbitrarily dense.

### 3.8 Computer Simulations

As part of the thesis, computer simulations were done using "Mathematica Software" to show the effectiveness of the proposed velocity control laws of the swarm model. The RK4 method was used to numerically integrate system (3.8) to confirm the emergent behavior of a sufficiently large number of individuals governed by the system. Extensive computer simulations show that for a sufficiently large number of individuals the proposed model (3.8) generates collective behaviors, some of which are similar to those reported in literature. Indeed, we shall utilize the same descriptions of the behaviors; however, in our case, we obtain them as a direct result of manipulating the cohesion parameters \( \gamma_i > 0, i \in \mathbb{N} \), which are a measure of the strength of attraction between an individual \( i \) and the swarm centroid, the coupling parameters \( \beta_{ij} > 0, i, j \in \mathbb{N}, i \neq j \), which are a measure of the strength of the interaction between individual \( i \) and individual \( j \), and the convergence parameters \( \alpha^s_i > 0, s = 1, 2; i \in \mathbb{N} \), which are a measure of the rate convergence of the \( i \)th individual to the swarm centroid.

#### 3.8.1 Scenario 1: Highly Parallel Group

Our first example (Figure 3.1) is an instance of a highly parallel group, described by Couzin et al. [19] as a group that self-organizes into a highly aligned arrangement with rectilinear motion. In our model, this behavior can be obtained if the cohesion parameters \( \gamma_i > 0, i \in \mathbb{N} \) are the same for all \( i \), the coupling parameters \( \beta_{ij} > 0, i, j \in \mathbb{N}, i \neq j \) are the same for all \( i \) and \( j \), and the convergence parameters \( \alpha^s_i > 0, s = 1, 2; i \in \mathbb{N} \) are the same for all \( i \). In other words, if our model is isotropic, then it can produce a highly parallel group.

Figures 3.2 and 3.3 shows the Lyapunov-like function and its time derivative along the system trajectory, respectively. Not only does the figure show that the conditions of practical stability have been satisfied but it also gives us the information where the swarms accelerated or decelerated. As shown in the figure, an increase in \( \dot{L}_i(x) \) indicates that the swarms is decelerating, whereas a decrease in \( \dot{L}_i(x) \) indicates that the swarms are accelerating.
Figure 3.1: A highly parallel group. There are $n = 30$ individuals (shown in red), each with bin size 10, randomly positioned at the initial time $t = 0$. The parameters are $\alpha_i^s = 1$, $s = 1, 2, 3$; and $\gamma_i = 2$ and $\beta_{ij} = 50$, $i, j \in \mathbb{N}$, $i \neq j$. The axes are $z_1(t)$, $z_2(t)$ and $z_3(t)$, respectively, for each individual $i$ at time $t \geq 0$. The initial positions of the individuals and the trace of their trajectories (grey) are shown. The green line shows the path of the centroid. As time evolves, they cluster around the centroid before moving in parallel with each other and in a straight line as a well-spaced cohesive group.
Figure 3.2: Scenario 1. Evolution of the Lyapunov-like function $L_i(x)$.

Figure 3.3: Scenario 1. The evolution of the time derivative of the Lyapunov-like function $\dot{L}_i(x)$.

Figure 3.4: Scenario 1. The velocity $v_2$ of the swarm.

Figure 3.5: Scenario 1. The velocity $w_2$ of the swarm.
3.8.2 Scenario 2: Oscillation about a Point

Our second example (Figure 3.8) shows a behavior that changes from a seemingly random one to an oscillating one about a point. In this emergent pattern, the swarm, as one cohesive group, moves in a circular motion. Interestingly, Forgoston and Schwartz [35] also reported a similar oscillating behavior; however this was teased out from a rotating swarm. In a rotating swarm, individuals form a ring about a point and circulate continuously along the ring, as in a rotating school of fish. The authors used a communication time delay on the rotating swarm, modeled as self-propelling individuals interacting through a pairwise attractive force in the presence of noise, to gradually provoke an oscillating behavior about the center of mass of the swarm. The evolution of the Lyapunov-like function, its time derivative and the velocity components have similar convergent behaviors as seen in Figures 3.2 to 3.7, for the rest of the scenarios that follows.
Figure 3.8: Oscillation about a point. There are \( n = 20 \) individuals (shown in red), each with bin size 10, randomly positioned at the initial time \( t = 0 \). The parameters are \( \alpha_i^s = 5, \ s = 1, 2, 3; \gamma_i = 5, \ i \in \mathbb{N} \); and \( \beta_{ij} > 0 \ (i, j \in \mathbb{N}, i \neq j) \) randomized between 300 and 500. The axes are \( z_1(t), z_2(t) \) and \( z_3(t) \), respectively, for each individual \( i \) at time \( t \geq 0 \). The initial positions of the individuals and the trace of their trajectories (grey) are shown. The green line shows the path of the centroid. After \( t = 0 \), the individuals wander aimlessly but always as a cohesive group. At some later time \( t > 0 \), the individuals become aligned with one another and perpetually oscillate clockwise around a point.
3.8.3 Scenario 3: Constant Arrangement about the Centroid

Our third example (Figure 3.9) shows a convergence to a constant arrangement about a stationary centroid. The same formation is the main outcome of the swarm model proposed by Gazi and Passino in 2003 [39]. In our model, this behavior can be obtained if the cohesion parameters ($\gamma_i > 0, i \in \mathbb{N}$) are the same for all $i$, the coupling parameters ($\beta_{ij} > 0, i, j \in \mathbb{N}, i \neq j$) are the same for all $i$ and $j$, and the convergence parameters ($\alpha_i^s > 0, s = 1, 2, 3; i \in \mathbb{N}$) are the same for all $i$.

Such a collective behavior could be seen in nature, for example, in myxobacterial fruiting-body formation, in which myxobacterial cells, when sensing starvation, change their movement pattern from outward spreading to inward concentration [131].
Figure 3.9: Convergence to a constant arrangement about the centroid. There are $n = 60$ individuals (shown in red), each with bin size 10, randomly positioned at the initial time of $t = 0$. The parameters are $\alpha_i^s = 0.1$, $s = 1, 2, 3$, and $\gamma_i = 0.2$ and $\beta_{ij} = 100$, $i, j \in \mathbb{N}$, $i \neq j$. The axes are $z_1(t)$, $z_2(t)$ and $z_3(t)$, respectively, for each individual $i$ at time $t \geq 0$. The trajectories are shown in grey. The centroid remains stationary.
3.8.4 Scenario 4: Leader-following Behavior

We expect an individual with a low cohesion parameter to be further away from a more compactly arranged group of individuals with more or less the same, but higher, cohesion parameters. Because of the effects of the attraction and the inter-individual collision-avoidance functions in the Lyapunov-like function, the individual with the lower cohesion parameter can either be following or leading the group. In this example, the cohesion parameters \( \gamma_i > 0, i \in \mathbb{N} \) are randomized between 0.01 and 1, and the coupling and convergence parameters are fixed. This means that some members can be further away from a more compact group of individuals. Figure 3.10 shows some members following a compact group in an aligned manner, and an outermost individual leading the group almost along the path of the centroid. We can assume that this is a leader-follower behavior.

Recently Justh and Krishnaprasad [57] and Morgan and Schwartz [91] proposed an individual-based continuum mechanics approach that utilizes the Frenet-Serret equations of motion to describe the position and orientation of interacting individuals in a swarm. Their models can be used to designate and control a leader, which then leads the swarm. The dynamics of their models – which result in an emergent behavior – depend on the initial conditions. Our approach differs in that the leader emerges from the swarm, and our system dynamics depend only on the system parameters, not on the initial conditions.
Figure 3.10: Leader-following behavior. There are $n = 30$ individuals (shown in red), each with bin size 10, randomly positioned at the initial time $t = 0$. The parameters are $\alpha_i^s = 1$, $s = 1, 2, 3$, and $\beta_{ij} = 30$. The cohesion parameters $\gamma_i$ are randomized between 0.01 and 1. The axes are $z_1(t)$, $z_2(t)$ and $z_3(t)$, respectively, for each individual $i$ at time $t \geq 0$. The grey lines show the trajectories and of the individuals. The path of the centroid is given by the green line.
3.8.5 Scenario 5: A Random Walk-like Phenomenon

In our fifth example, we encounter an interesting behavior that is very similar to a random walk phenomenon. A random walk is a random process consisting of a sequence of discrete steps of fixed length [152]. Using our model, this behavior can be induced by allocating large values of the coupling parameter to each individual. The simulations show a cohesive group with individuals hovering excitedly about the centroid in a random fashion. As they change positions, the centroid traces out a series of straight segments interrupted by tight turns.

Random walk-based models have been successfully used to model swarming behavior. For example, Majkut [85] modeled the flight paths of fruit flies *Drosophila melanogaster*, which utilize scent to locate food sources in their vicinity. Fruit fly flight is characterized by a series of straight segments interrupted by rapid changes in horizontal heading known as *saccades*. Majkut used *Levy flights* to model the foraging behavior of fruit flies. Levy flights are a class of continuous time random walks, which are often found in biological behavior and are prevalent in foraging.
Figure 3.11: A random walk-like behavior. There are $n = 30$ individuals (shown in red), each with bin size 10, randomly positioned at the initial time $t = 0$. The parameters are $\alpha_i^s = 5$, $s = 1, 2, 3$, and $\gamma_i = 1$. The coupling parameters $\beta_{ij}$ are randomized between and including 200 and 500. The axes are $z_1(t)$, $z_2(t)$ and $z_3(t)$, respectively, for each individual $i$ at time $t \geq 0$. The grey lines shows the trajectories of the individuals. The path of the centroid is shown thick in green. The swarm is cohesive throughout.
3.8.6 Scenario 6: A Spiral-Like Behavior

In our sixth example, we encounter an interesting behavior that is very similar to a spiral behavior. A spiral is a curve which emanates from a central point, getting progressively farther away as it revolves around the point. Using our model, this behavior can be induced by allocating large randomized values of the coupling parameter to each individual. The simulations show a cohesive group with individuals hovering excitedly about the centroid in a spiral fashion. As they change positions, the centroid traces out spiral curves.

Many millipedes defend themselves by rolling their bodies up into a ball or spiral. This behavior protects the legs and delicate underside of the animal, leaving only the hard plates of the body segments exposed. At the beginning phase, the swarm members gradually aggregate and form a cohesive cluster. Then, they continuously move in the same direction as a group, and eventually evolve into a spiral motion.
Figure 3.12: A spiral-like behavior. There are \( n = 20 \) individuals (shown in red), each with bin size 10, randomly positioned at the initial time \( t = 0 \). The parameters are \( \alpha_i^s = 5 \), \( s = 1, 2, 3 \), and \( \gamma_i^s = 5 \). The coupling parameters \( \beta_{ij} \) are randomized between and including 300 and 500. The axes are \( z_1(t) \), \( z_2(t) \) and \( z_3(t) \), respectively, for each individual \( i \) at time \( t \geq 0 \). The grey lines show the trajectories of the individuals. The path of the centroid is shown thick in green. The swarm is cohesive throughout.
3.9 Concluding Remarks

This chapter displays the set of continuous, time-invariant velocity control laws, derived from the Lyapunov-based control scheme to show the emergent behavior arising from a swarm of boids. This, has been treated for the first time in literature, via the Lyapunov-based control scheme to construct Lyapunov-like functions. The different emerging behaviors was a result of varying the control parameters in each of the case. The efficiency of the control laws have been demonstrated through exigent and interesting simulations arising from the emergent behaviors of the swarm. This showed that the swarm model is a gradient system that is practically stable about the centroid.

The next chapter will attempt to extend the results of this chapter and focus on the generalised continuous time-invariant controllers for swarm of boids via the Lyapunov-based control scheme. The qualities of this chapter include introduction of the stationary obstacles, due to its existence in the real world, and controllers that will mimic the emergent swarm behaviors. The practical stability of the system will be proved.
Practical Stability Analysis via Lyapunov-like Function in the Presence of Obstacles

"Practical application is found by not looking for it, and one can say that the whole progress of civilization rests on that principle."

Jacques Salomen Hadamard (1865 - 1963)

4.1 Introduction

Motion planning and control for swarms have appeared constantly in literature. This is in relation to its effectiveness due to the exhibition and coupling of inherent constraints and restrictions, its wide-ranging capabilities and abundance of real world applications [118]. These are essential for accomplishing specific control objectives. A few prominent applications of swarming research include formation flight control, surveillance, reconnaissance, construction, warehouse automation, transportation, healthcare, mining, cooperative manipulation, search and rescue missions and planetary exploration [114]. These applications invariably demand a high rate of system effectiveness [3]. Thus, this requirement is one of the main motivations of employing multi-agents. in comparison with a single agent, multi-agents can perform an allocated task more efficiently in terms of time and quality, harness desired behaviors, easily accomplish tasks not executable by a single robot, enhance performance and robustness, easier and cheaper and has the benefits of highly distributed sensing and actuation [118].

This chapter considers the emergent behavior arising from the swarm model when obstacles are introduced into the workspace. That is, we construct a Lyapunov-like function via the LbCS that guarantees the emergent behavior arising from the swarm, considering all practical limitations and constraints due to fixed obstacles.

The remainder of this chapter is organized as follows: in Section 4.2, the attractive and
repulsive potential field functions are designed in accordance with the LbCS; in Section 4.3, the velocity controllers are designed and stability analysis of the system carried out; in Section 4.4, we illustrate the feasibility of the proposed controllers via computer simulations of the emergent swarm behaviors; and finally, Section 4.5 closes the chapter with a brief conclusion.

4.2 Deployment of Lyapunov-based Control Scheme

Again, the principal objective is to construct artificial potential functions (APFs), that is, for this case, a Lyapunov-like function from a recently developed Lyapunov-based control scheme (LbCS) found in [114] and as per the procedure outlined in Table 2.1, and accordingly, derive the velocity controls, $v_i$, $w_i$, and $u_i$, such that the swarm of boids will be able to exhibit swarming behavior in some certain direction whilst avoiding obstacles. The control scheme appropriately combines these APFs to form a Lyapunov-like function candidate – a platform to design the nonlinear velocity controllers for the swarm of boids. This Lyapunov-like function candidate will also be utilized in a later section to prove stability of the system.

A dichotomy of APFs will be designed in the following subsections, that is, we construct attractive functions and obstacle avoidance functions for the attraction to the centroid and the repulsion from the various obstacles, respectively.

4.2.1 Attraction to the Centroid

To ensure that the individuals of the swarm are attracted towards each other and also form a cohesive group by having a measurement of the distance from the $i$th individual to the swarm centroid, we use the following function from the previous chapter, equation (3.4):

$$R_i(x) := \frac{1}{2} \left[ \left( x_i - \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 + \left( y_i - \frac{1}{n} \sum_{i=1}^{n} y_i \right)^2 + \left( z_i - \frac{1}{n} \sum_{i=1}^{n} z_i \right)^2 \right], \quad i \in \mathbb{N}.$$

This will be part of a Lyapunov-like function for system (3.3) which ensures that $i$th individual is attracted to the swarm centroid.

4.2.2 Inter-individual Collision Avoidance

With equation (3.1) of the $i$th individual in mind and for the boids to avoid each other, we use the following function from the previous chapter, equation (3.5):

$$Q_{ij}(x) := \frac{1}{2} \left[ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (r_i + r_j)^2 \right].$$

The function is an Euclidean measure of the distance between the individual boids, and will appear in the denominator of an appropriate term in the candidate Lyapunov-like function to be proposed.
4.2.3 Swarming in the Presence of Fixed Obstacles

If a swarm encountered an obstacle in its path, how would it behave? Nature provides instances of the resultant behaviors – a flock of bird may split and then rejoin [109]; a swarm of zooplankton *Daphnia magna* may swirl about a marker [100, 92]; a bacterial swarm may increase their density in the presence of antibiotics [11].

Now, we aim to show that it is straightforward to extend our model (3.8) to include fixed obstacles.

We now fix a set of solid objects fixed within the boundaries of the workspace and define an obstacle space for the stationary solid object. Since these objects can be of any shape (regular or irregular), first a plan will need to be devised to ensure that the entire body of an object is enveloped within the defined obstacle space. To construct a suitable obstacle space for each stationary object, we shall adopt the methodology given in [128], whereby the solid objects need to be represented as simpler fixed-shaped objects such as a circle, a polygon or a convex hull [114, 126, 127].

It can be verified geometrically that the most simplest and convenient fixed-shaped object in the Euclidean plane is a disk [114]. Hence, we shall consider disk-shaped obstacles as one category of stationary solid objects. But since we are considering a three-dimensional space, we shall consider spherical-shaped obstacles as the first category of stationary solid objects. Although spheres help capture most of the obstacles in the workspace, the control scheme utilized to construct obstacle avoidance functions works well for solid objects that are somewhat spherical-shaped.

Other classes of obstacles includes rod-shaped obstacles and elliptic-shaped obstacles. These are classed as stationary obstacles. Moreover, there are moving obstacles as well like the blind-man [114]. As a matter of clarity and simplicity, we shall confine ourselves to spherical-shaped obstacles to illustrates the effectiveness of the velocity controllers in this thesis.

For the purpose of illustrating our method, we consider the simplest obstacle, namely, a sphere, which is a convex obstacle.

Let there be \( m \in \mathbb{N} \) fixed obstacles within the boundary of the workspace. We have the following definition:

**Definition 4.2.1.** The kth spherical-shaped obstacle is centered at \((o_{k1}, o_{k2}, o_{k3})\), \(k = 1, \ldots, m\), with radii \( r_{ok} > 0 \). Precisely, the kth disk-shaped obstacle is the set

\[
o_k := \{(z_1, z_2, z_3) \in \mathbb{R}^3 : (z_1 - o_{k1})^2 + (z_2 - o_{k2})^2 + (z_3 - o_{k3})^2 \leq r_{ok}^2 \}\.
\]
For the avoidance of these fixed obstacles, we consider the following obstacle avoidance function:

\[ W_{ik}(x) := \frac{1}{2} \left[ (x_i - o_{k1})^2 + (y_i - o_{k2})^2 + (z_i - o_{k3})^2 - (r_i + r_{ok})^2 \right], \]

for \( i = 1, \ldots, n \) and \( k = 1, \ldots, m \). The function \( W_{ik}(x) \) is a measurement of the distance between the \( i \)th individual and \( k \)th obstacle \( o_k \).

### 4.3 Design of the velocity controllers and the Stability Analysis

The nonlinear control laws for system (3.2) will be designed using the LbCS. In parallel, the control scheme will then utilize Theorem 3.2.1 to provide the mathematical proof of the practical stability of the system (3.2).

#### 4.3.1 Lyapunov-like Function

As per the LbCS, we combine all the attractive and the repulsive potential field functions, introduce tuning parameters (or control parameters). Given some parameter \( \gamma_i > 0 \), \( \beta_{ij} > 0 \) and \( \omega_{ik} > 0 \) \( i, j, k \in \mathbb{N} \), we define a Lyapunov-like function for system (3.2) as

\[
V_i(x) = L_i + \sum_{k=1}^{m} \frac{\omega_{ik} R_i(x)}{W_{ik}(x)}
\]

\[
= \gamma_i R_i(x) + \sum_{j=1, j \neq i}^{n} \beta_{ij} R_i(x) \frac{Q_{ij}(x)}{W_{ik}(x)} + \sum_{k=1}^{m} \frac{\omega_{ik} R_i(x)}{W_{ik}(x)},
\]

so that our new Lyapunov-like function for system (3.3) now becomes

\[ V(x) := \sum_{i=1}^{n} V_i(x_i), \]

which is clearly continuous and locally positive definite on the domain

\[ E(V) := \left\{ x \in \mathbb{R}^{2n} : \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} Q_{ij}(x) > 0 \text{ and } \sum_{i=1}^{n} \sum_{k=1}^{m} W_{ik}(x) > 0 \right\}, \]

noting that \( x^* \notin E(V) \).

Assume next that the ratios are added appropriately to the Total Potential Field. Then any increase in a ratio cannot be unbounded because the existence of the Lyapunov function
over an appropriate domain implies the boundedness of the state trajectories over the domain corresponding to any bounded initial condition within the domain. This means that the artificial potential field generated by $V(x)$ will not allow the swarms to get too close or collide with the obstacles. Note that any increase in the above ratios does not translate to an increase in $V \equiv V(t)$, simply because $V$, by its nature, is non-increasing in time $t \geq 0$ and any increase in one of the ratios gives a smaller or the same value of $V$ at time $t$ compared to all previous values of $V$.

As such, the essence of obstacle avoidance capability in the LbCS lies, therefore, in the creation of obstacle avoidance functions that will induce an increase or decrease in the instantaneous rate of change of the tentative Lyapunov-like function. The reader is referred to [114] for a detailed anecdote of the effects of the obstacle avoidance functions and the resulting repulsive potential field functions.

4.3.2 Nonlinear Velocity Controllers for the Swarm

Now, carrying out the same analysis as in subsection 3.4.1, the time-derivative of $V$ along every solution of system (3.3) is the dot product of the gradient of $V$, given by,

$$\nabla V = \left( \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial y_1}, \frac{\partial V}{\partial z_1}, \ldots, \frac{\partial V}{\partial x_n}, \frac{\partial V}{\partial y_n}, \frac{\partial V}{\partial z_n} \right),$$

and the time-derivative of the state vector $x = (x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$. That is,

$$\dot{V}(x) = \nabla V(x) \cdot \dot{x}$$

$$= \sum_{i=1}^{n} \gamma_i \dot{R}_i(x) + \sum_{j=1}^{n} \sum_{j \neq i}^{n} \beta_{ij} \dot{Q}_{ij}(x) \dot{R}_i(x) - \sum_{j=1}^{n} \sum_{j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} \dot{Q}_{ij}(x)$$

$$+ \sum_{k=1}^{m} \frac{\omega_{ik} \dot{R}_i(x)}{W_{ik}(x)} - \sum_{k=1}^{m} \frac{\omega_{ik} R_i(x)}{W_{ik}(x) \dot{W}_{ik}(x)},$$

where

$$\sum_{i=1}^{n} \dot{R}_i(x) = \sum_{i=1}^{n} \left[ \left( x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - \frac{1}{n} \sum_{m=1}^{n} \left( x_m - \frac{1}{n} \sum_{k=1}^{n} x_k \right) \right] x_i'$$

$$+ \sum_{i=1}^{n} \left[ \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - \frac{1}{n} \sum_{m=1}^{n} \left( y_m - \frac{1}{n} \sum_{k=1}^{n} y_k \right) \right] y_i'$$

$$+ \sum_{i=1}^{n} \left[ \left( z_i - \frac{1}{n} \sum_{k=1}^{n} z_k \right) - \frac{1}{n} \sum_{m=1}^{n} \left( z_m - \frac{1}{n} \sum_{k=1}^{n} z_k \right) \right] z_i',$$
\[\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \dot{Q}_{ij}(x) = 2 \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} (x_i - x_j)x'_i + 2 \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} (y_i - y_j)y'_i + 2 \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} (z_i - z_j)z'_i\]

and
\[\sum_{i=1}^{n} \sum_{k=1}^{m} \dot{W}_{ik}(x) = \sum_{i=1}^{n} \sum_{k=1}^{m} (x_i - o_{k1})x'_i + \sum_{i=1}^{n} \sum_{k=1}^{m} (y_i - o_{k2})y'_i + \sum_{i=1}^{n} \sum_{k=1}^{m} (z_i - o_{k3})z'_i.\]

Noting that \(\frac{1}{n} \sum_{m=1}^{n} \left( h_m - \frac{1}{n} \sum_{k=1}^{n} h_k \right) = 0\) for any \(h_i \in \mathbb{R}, i = 1, 2, \ldots, n\), we simplify the former expression to
\[\sum_{i=1}^{n} \dot{R}_i(x) = \sum_{i=1}^{n} \left[ (x_i - \frac{1}{n} \sum_{k=1}^{n} x_k) x'_i + \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) y'_i + \left( z_i - \frac{1}{n} \sum_{k=1}^{n} z_k \right) z'_i \right].\]

Now, collecting terms with \(x'_i, y'_i\) and \(z'_i\), and substituting \(x'_i = \dot{x}_i = v_i, y'_i = \dot{y}_i = w_i\) and \(z'_i = \dot{z}_i = u_i\) from system (3.2), we have
\[\dot{V}(x) = \sum_{i=1}^{n} \left\{ \left( \gamma_i + \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - 2 \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (x_i - x_j) \right\} \dot{x}_i\]
\[+ \sum_{i=1}^{n} \left\{ \left( \gamma_i + \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - 2 \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (y_i - y_j) \right\} \dot{y}_i\]
\[+ \sum_{i=1}^{n} \left\{ \left( \gamma_i + \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( z_i - \frac{1}{n} \sum_{k=1}^{n} z_k \right) - 2 \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (z_i - z_j) \right\} \dot{z}_i\]
\[= \sum_{i=1}^{n} \left[ \frac{\partial V}{\partial x_i} \cdot \dot{x}_i + \frac{\partial V}{\partial y_i} \cdot \dot{y}_i + \frac{\partial V}{\partial z_i} \cdot \dot{z}_i \right]\]
\[= \sum_{i=1}^{n} \left[ \frac{\partial V}{\partial x_i} \cdot v_i + \frac{\partial V}{\partial y_i} \cdot w_i + \frac{\partial V}{\partial z_i} \cdot u_i \right].\]
Let there be real numbers $\mu_i > 0$, $\nu_i > 0$ and $\eta_i > 0$ such that

$$v_i = -\mu_i \frac{\partial V}{\partial x_i}, \quad w_i = -\nu_i \frac{\partial V}{\partial y_i} \quad \text{and} \quad u_i = -\eta_i \frac{\partial V}{\partial z_i}.$$  

Then

$$\dot{V}(x) = -\sum_{i=1}^{n} \left[ \mu_i \left( \frac{\partial V}{\partial x_i} \right)^2 + \nu_i \left( \frac{\partial V}{\partial y_i} \right)^2 + \eta_i \left( \frac{\partial V}{\partial z_i} \right)^2 \right]$$

$$= -\sum_{i=1}^{n} \left[ \frac{v_i^2}{\mu_i} + \frac{w_i^2}{\nu_i} + \frac{u_i^2}{\eta_i} \right] \leq 0,$$

for all $x \in D(V)$.

For the $i\text{th}$ individual, system (3.2) therefore becomes

$$\begin{aligned}
x_i'(t) &= v_i(t) = v_i(x(t)) = -\mu_i \frac{\partial L}{\partial x_i}, \\
y_i'(t) &= w_i(t) = w_i(x(t)) = -\nu_i \frac{\partial L}{\partial y_i}, \\
z_i'(t) &= u_i(t) = u_i(x(t)) = -\eta_i \frac{\partial L}{\partial z_i},
\end{aligned}$$

(4.1)

$$x_i(t_0) = x_i(0), \quad y_i(t_0) = y_i(0), \quad z_i(t_0), \quad t_0 \geq 0,$$

where

$$\frac{\partial V}{\partial x_i} = \left( \gamma_i + \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - 2 \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (x_i - x_j) - \sum_{k=1}^{m} \frac{\omega_{ik} R_i(x)}{W_{ik}^2(x)} (x_i - o_{k1}),$$

and

$$\frac{\partial V}{\partial y_i} = \left( \gamma_i + \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - 2 \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (y_i - y_j) - \sum_{k=1}^{m} \frac{\omega_{ik} R_i(x)}{W_{ik}^2(x)} (y_i - o_{k2}),$$

and

$$\frac{\partial V}{\partial z_i} = \left( \gamma_i + \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} \right) \left( z_i - \frac{1}{n} \sum_{k=1}^{n} z_k \right) - 2 \sum_{j=1, j\neq i}^{n} \frac{\beta_{ij} R_i(x)}{Q_{ij}^2(x)} (z_i - z_j) - \sum_{k=1}^{m} \frac{\omega_{ik} R_i(x)}{W_{ik}^2(x)} (z_i - o_{k3}).$$
Then system (3.3) becomes the new gradient system
\[
\dot{x} = G(x) = -H(\nabla V(x)), \quad x_0 := x(t_0), \quad t_0 \geq 0,
\]
(4.2)
It is clear that \( G \in C[E(V), \mathbb{R}^n] \).

### 4.3.3 Practical Stability Analysis

In this subsection, we shall prove the practical stability of system (3.3). Using the same method of analysis as in subsection 3.5, we will constructively use the method by Lakshmikantham, Leela and Martynyuk [69].

**Theorem 4.3.1.** System (4.2) is uniformly practically stable.

**Proof.** The proof is very similar to section 3.5.

### 4.4 Computer Simulations

#### 4.4.1 Scenario 1

As our first example in this section, we mimic the situation where the swarm converges to an obstacle and swirls endlessly about it. We consider 20 individuals, each with bin size 20, randomly placed initially into the workspace. There are 20 spherical obstacles of radius 100 placed on top of each other to deduce a cylindrical-type of obstacle. Figure 4.1 shows the swarm (shown in red) moving about a field of cylindrical-type of obstacle.

To illustrate the convergent property of the control laws, graphs of the velocity components of the first boid have been generated. The corresponding graphs of the remaining boids will exhibit similar convergent properties along the system trajectories.
Practical Stability Analysis in the Presence of Obstacles

(a) Initial state of the swarm ($t = 0$).
(b) State of the swarm at $t = 100$ units.
(c) State of the swarm at $t = 300$ units.
(d) State of the swarm at $t = 1000$ units.

Figure 4.1: Compactness and mobility for obstacle avoidance. The parameters are $\alpha_i^s = 1$, $i = 1, \ldots, 20$, $s = 1, 2, 3$, $\gamma_i$ randomized between 0.5 and 1, $\beta_{ij} > 0$, $i, j \in \mathbb{N}$, $i \neq j$ randomized between 10 and 15 and $\omega_{ik} = 1$. The axes are $z_1(t)$, $z_2(t)$ and $z_3(t)$, respectively, for each individual $i$ at time $t \geq 0$. The initial positions and the onset of swarming are shown in (a). In (b) and (c), the individuals (shown in red) congregate and group to swirl past the obstacle. In (d), the swarm settles into a cruise formation. The trajectories are shown in grey. The centroid is shown by the green color.
Practical Stability Analysis in the Presence of Obstacles

(a) Initial state of the swarm ($t = 0$).

(b) State of the swarm at $t = 100$ units.

(c) State of the swarm at $t = 700$ units.

(d) State of the swarm at $t = 900$ units.

Figure 4.2: The top view of the swarm whilst avoiding the cylindrical type obstacles.
Figure 4.3: *Scenario 1.* Evolution of the Lyapunov-like function $L_i(x)$.

Figure 4.4: *Scenario 1.* The evolution of the time derivative of the Lyapunov-like function $\dot{L}_i(x)$.

Figure 4.5: *Scenario 1.* The velocity $v_1$ of the swarm.

Figure 4.6: *Scenario 1.* The velocity $w_1$ of the swarm.
4.4.2 Scenario 2

As our second example in this section, we look at the situation where the swarm converges to a cylindrical obstacle, avoiding it and exhibit a spiral-like behavior. This could be modeled as a flock of boids moving happily in a spiral-like cruise formation. We consider 20 individuals, each with bin size 20, randomly placed initially into the workspace. There are 20 spherical obstacles of radius 100 placed on top of each other to deduce a cylindrical-type of obstacle. Figure 4.9 shows the swarm (shown in red) moving about a field of cylindrical-type of obstacle.
Figure 4.9: Compactness and mobility for obstacle avoidance. The parameters are $\alpha^s_i$, $i = 1, \ldots, 20$, $s = 1, 2, 3$, randomized between 0.5 and 1, $\gamma_i$ randomized between 0.5 and 1, $\beta_{ij} > 0$, $i, j \in \mathbb{N}$, $i \neq j$ randomized between 100 and 150 and $\omega_{ik} = 0.5$. The axes are $z_1(t)$, $z_2(t)$ and $z_3(t)$, respectively, for each individual $i$ at time $t \geq 0$. The initial positions and the onset of swarming are shown in (a). In (b) and (c), the individuals (shown in red) congregate and group up and spiral past the obstacles. In (d), the swarm settles into a spiral formation. The trajectories are shown in grey. The centroid is shown by the green color.
Figure 4.10: The top view of the swarm whilst avoiding the cylindrical type obstacles.
4.4.3 Scenario 3

As our third example in this section, we look at the situation where the swarm converges to a cylindrical obstacle and exhibit a random walk-like behavior. We consider 20 individuals, each with bin size 20, randomly placed initially into the workspace. There are 50 spherical obstacles of radius 100 placed on top of each other to deduce a cylindrical-type of obstacle. Figure 4.12 shows the swarm (shown in red) moving about a field of cylindrical-type of obstacle. The cohesion parameters play a big role in inducing this particular emergent behavior.
Figure 4.11: Compactness and mobility for obstacle avoidance. The parameters are $\alpha_i^s$, $i = 1, \ldots, 20, s = 1, 2, 3$, randomized between 0.5 and 1, $\gamma_i$ randomized between 0.5 and 1, $\beta_{ij} > 0$, $i, j \in \mathbb{N}$, $i \neq j$ randomized between 100 and 150 and $\omega_{ik} = 0.5$. The axes are $z_1(t)$, $z_2(t)$ and $z_3(t)$, respectively, for each individual $i$ at time $t \geq 0$. The initial positions and the onset of swarming are shown in (a). In (b) and (c), the individuals (shown in red) congregate and group to swirl past the obstacle. In (d), the swarm settles into a cruise formation. The trajectories are shown in grey. The centroid is shown by the green color.
Practical Stability Analysis in the Presence of Obstacles

Figure 4.12: The top view of the swarm whilst avoiding the cylindrical type obstacles.
4.5 Concluding Remarks

Our proposed model is essentially a distributed control system wherein each individual has its own controller that controls its position and instantaneous velocity. Ultimately, the design of a distributed control system for the emergent swarm behaviors is a complex, computer intensive, yet an interesting problem. The control system used in this chapter is not scalable. In this chapter, we have presented a set of distributed continuous time-invariant velocity control laws that results in the emergent swarm behaviors while avoiding obstacles in a constrained environment. The milestone of this chapter was the introduction of spherical shaped obstacle into the workspace. The generalized controllers, extracted from the Lyapunov-based control scheme (LbCS), enabled collision free trajectories of the swarm within a constrained environment, whilst satisfying the intimately couple holonomic constraints of the swarm, and the kinodynamic constraints associated with the system. The effectiveness of the proposed control laws was demonstrated via computer simulations of different emerging behaviors.

In the next chapter, we will deploy the LbCS to derive continuous time-invariant control laws to address the problem of motion planning and control of a swarm of boids within an obstacle-ridden but dynamic environment. The dynamic obstacles includes a swarm of point masses and car-like robots. The resulting controllers will generate and guarantee the execution of an array of sub-tasks, which include: goal convergence, team coordination and cohesion, and adherence to nonholonomic and kinodynamic constraints. Furthermore, we will introduce a new recipe, which is an introduction of a two-dimensional swarm, within the framework of the LbCS.
Swarm Navigation in a Dynamic Environment

"I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our "creations," are simply the notes of our observations."

Godfrey Harold Hardy (1877 - 1947)

5.1 Introduction

Trajectory Planning and control of holonomic and nonholonomic systems has been an active area of research for more than two decades now. Basically, it involves finding a feasible trajectory from some initial configuration to a desired one while satisfying the velocity constraints of the system. In recent years, with the rapid advances in sensing, communication, computation, and actuation capabilities, groups or swarms are expected to cooperatively perform dangerous or explorative tasks in a broad range of potential applications. As highlighted by Latombe [72], motion planning is “eminently necessary, since, by definition, a robot accomplishes tasks by moving in the real world”. The essence of robot motion planning problem can be formulated as a two-dimensional problem and is captured in the following classic definition (adopted from [114]):

Definition 5.1.1. Given a robot and a description of its workspace, propose a path that the robot can follow. In particular, if the workspace is cluttered with solid objects, propose a collision-free path that can lead the mobile robot from the desired starting point to the desired goal or target.

Sharma [114] described that a comprehensive solution to the findpath problem must take into account the generation of a smooth, collision-free path that ensures that the mobile robot reaches its target in a reasonable amount of time. When this additional constraint is integrated to the findpath problem, it yields the piano mover’s problem [72]. The three generic
motion planning paradigms for mobile robots for operating in an environment cluttered with static and dynamic obstacles are:

1. Road map – consists of capturing the connectivity of the robot’s free space, that is, a set of routes connecting and linking the initial and final configurations as identified within the robot’s free space;
2. Cell decomposition – the robot’s free path between the initial configuration and the final configuration is decomposed into smaller regions, called free cells, so that a path between any two configurations in a cell is easily generated;
3. Potential field – involves modeling the robot as a point moving under the influence of an artificial potential field that is determined by the set of obstacles and the target configuration. The findpath problem is reformulated numerically by imposing a mathematical function over the robot’s configuration space [72].

Devising motion planning algorithms for multi-agents sharing a common workspace is inherently difficult. This is a result of the environment being no longer static but dynamic. Static environments have provided excellent breeding grounds for high-powered algorithms so far [4, 62]. However, more recently there has been a shift of emphasis to include dynamic environments due to its applications in the real world. The dynamic environment is composed of both the stationary and the unpredictable (or predictable) dynamic obstacles [118]. These dynamic obstacles can incorporate the mobile robots themselves as well as other moving solid objects or obstacles in the environment. Thus, fundamental to the motion planning problem of multi-agents is the need to control and plan the motions of the agents that would yield inter-agent and agent to obstacle collision avoidances. Numerous papers have discussed this problem, some of which includes methods such as discretization of the configuration time-space using sequential space slicing [32], sheared cylindrical representations of moving obstacles and generating optimal tangential paths to the goals [105], hybrid systems [30], threaded petri nets [63], plan-merging [1], negotiations [45], online artificial potential fields strategy [64, 72], decomposition of the problem into path planning and velocity planning sub-problems [58] and a Lyapunov-based control scheme for various nonholonomic multi-agents [114, 118, 119], to name a few.

This chapter explores the challenging but indispensable area of multi-agent research. It will focus on generalizing the control laws for multi-vehicle system within the Lyapunov-based control scheme (LbCS). In addition to considering multiple vehicles and dynamic environments, the other novel aspects of this chapter with are moving obstacles. Multiple car-like robots will constitute this category of dynamic obstacles. Hence, there will be a number of car-like robots moving between start and goal configurations in a constrained environment. The moving obstacles of a car-like robot will be all the other car-like robots in an a priori known workspace cluttered with static obstacles or moving obstacles. The other novel novel aspect would be the 2-dimensional swarm introduced into the workspace avoiding the car-like robots.

The remainder of the chapter is organized as follows: in Section 5.2, the car-like vehicle is described and the 2-dimensional swarm model is formulated; in Section 5.3, the attractive and
repulsive potential field functions are designed in accordance with the LbCS to address the
motion planning and control problem of multi-vehicle system; in Section 5.4, the acceleration
collectors for the car-like mobile robots and the velocity controllers for the swarm of boids
are designed and stability analysis is carried out; in Section 5.6, computer simulations of three
scenarios are carried out; in Section 5.7, the repulsive functions for the obstacle avoidance are
constructed; in Section , the acceleration collectors for the car-like mobile robots and the
velocity collectors for the swarm of boids are designed and stability analysis is carried out; in
Section 5.10, computer simulations of two scenarios are carried out and in Section 5.11 closes
the chapter with a brief conclusion.

5.2 System Modeling

In this section, we shall model a rear driven car-like vehicle and a general 2-dimensional
swarm. Both the models will be used to illustrate via the Lyapunov-based control scheme the
effectiveness of the system models. Our problem statement is

Given a set of fixed and moving obstacles (swarm of boids) in the $z_1z_2$-plane, design
the controllers $\sigma_{k1}$ and $\sigma_{k2}$ so that the robot can navigate via these obstacles and
finally reach the target. Firstly, the robot will navigate in the workspace with only the
swarm of boids as moving obstacles. Moreover, if the workspace contains multiple
fixed obstacles scattered randomly, then describe or design an algorithm which can
avoid these obstacles and the swarm of boids whereby the swarm also avoids the
obstacles that can be used to guide the robot to its target.

5.2.1 Vehicle Model - The Kinematics and Dynamics of the Car-like Robot

In this subsection, the kinematics and the dynamics of a car-like system will be described.
The vehicle model consists of a rear wheel driven car-like vehicle, whereby engine power is
applied to the rear wheels (see Figure 5.1). Although polar coordinates are more popular
with moving obstacle [66], we utilize the Cartesian coordinate system since it does not inject
undesired singularities into the navigation problem [114].

**Definition 5.2.1.** The $k$th nonholonomic car-like mobile robot is a circular disk with $rv_k$ and
is positioned at center $(xv_k, yv_k)$. In addition, the $k$th car-like robot is the set

$$A_k = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - xv_k)^2 + (z_2 - yv_k)^2 \leq rv_k\}, \text{ for } k \in \{1, ..., m\}, m \in \mathbb{N},$$

where $A_k$ embodies a rear-wheel driven and front-wheel steered car-like vehicle.

To accommodate dynamics of the system, we include the acceleration components as well.
Thus, if we let $m_k$ be the mass of the car-like robots, $F_k$ the force along the axis of the car-like
robots, $\Gamma_k$ the torque about a vertical axis at $(xv_k, yv_k)$ and $I_k$ the moment of inertia of the
vehicle, then the vehicle system is a dynamic model. If $L$ is the distance between the two
Figure 5.1: A rear wheel driven vehicle with front wheel steering and steering angle $\phi_k$. axles and $l$ the length of each axle, then the dynamic model of the car-like vehicle is
\[
\begin{align*}
\dot{x}_v^k &= v_k \cos \theta_k - \frac{L}{2} \omega_k \sin \theta_k, \\
\dot{y}_v^k &= v_k \sin \theta_k + \frac{L}{2} \omega_k \cos \theta_k, \\
\dot{\theta}_k &= \omega_k, \\
\dot{v}_k &= \sigma_{k1} := F_k / m_k, \\
\dot{\omega}_k &= \sigma_{k2} := \Gamma_k / I_k,
\end{align*}
\tag{5.1}
\]
where the variable $\theta_k$ gives the car’s orientation with respect to the main axes, $v_k$ and $\omega_k$ are the translational and rotational velocities, respectively, while $\sigma_{k1}$ and $\sigma_{k2}$ are, respectively, the instantaneous translational and rotational accelerations. Hereafter, we shall use the vector notation $z = (x_v^k, y_v^k, \theta_k, v_k, \omega_k) \in \mathbb{R}^5$ to describe the variables in (5.1).

Referring to Figure 5.1, to ensure that the entire vehicle safely steers past an obstacle, the planar vehicle will be enclosed by the smallest circle possible. $L$ and $l$ are, respectively, the length and width of the vehicle, then given the clearance parameters (safety parameters) $\epsilon_1$ and $\epsilon_2$, enclose the vehicle by a protective circular region centered at $(x_v^k, y_v^k)$, with radius $r v_k := \frac{1}{2} \sqrt{(2\epsilon_1 + L)^2 + (2\epsilon_2 + l)^2}$. By doing this, essentially a well-known technique in mobile robot path-planning schemes is followed wherein the robot is represented as a simpler fixed-shaped object, such as a circle, a polygon or a convex hull [128].

**Assumption 5.2.1.** The instantaneous accelerations $\sigma_{k1}$ and $\sigma_{k2}$ can move the car-like robot of $A_k$ to its designated target and attain the desired final orientation.

**Remark 5.2.1.** By the Lyapunov-based control scheme, $\sigma_{k1}$ and $\sigma_{k2}$ is considered as the set of nonlinear acceleration controllers of $A_k$, for $k \in \{1, 2, \ldots, n\}$. Since $A_k$ has knowledge of
the position of all $A_j$, for $j = 1, \ldots, n$, $j \neq k$, but knowledge of its target only, the controllers are duly decentralized in nature.

5.2.2 A Two Dimensional Swarm Model

Following the nomenclature of Reynolds [109], each member of the flock is denoted as a boid. A general swarm model, that was adopted in chapters 3 and 4, formulated by Mogilner et al. [89] will be utilized. We shall consider a swarm of $m$ individuals moving with the velocity of the swarm’s centroid.

At time $t \geq 0$, let $(x_b(t), y_b(t)), i = 1, \ldots, n$, be the planar position of the $i$th individual, which we shall define as a point mass residing in a disk of radius $r_b \geq 0$,

$$B_i = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - x_b(t))^2 + (z_2 - y_b(t))^2 \leq r_b^2\}.$$ (5.2)

Let us define the centroid of the swarm as

$$(x_b, y_b) = \left(\frac{1}{m} \sum_{i=1}^{m} x_b, \frac{1}{m} \sum_{i=1}^{m} y_b\right).$$ (5.3)

At time $t \geq 0$, let $(v_b(t), \omega_b(t)) := (\dot{x}_b(t), \dot{y}_b(t))$ be its instantaneous velocity of the $i$th point mass. Using the above notations, we thus have a system of first order ODE’s for the $i$th individual, assuming the initial conditions at $t = t_0 \geq 0$:

$$\begin{align*}
\dot{x}_b &= v_b(t), \\
\dot{y}_b &= \omega_b(t), \\
x_{b0} &:= x_b(t_0), \; y_{b0} := y_b(t_0).
\end{align*}$$ (5.4)

5.3 Deployment of the Lyapunov-based Control Scheme

The principal control objective of this section is to utilize the Lyapunov-based control scheme to design the translational acceleration $\sigma_{k1}$ and the rotational acceleration $\sigma_{k2}$ such that the car-like robot, represented by system (5.1), will navigate safely among obstacles, reach a neighborhood of its destination or target whilst respecting kinodynamic constraints and be aligned to a prescribed final posture.

In accordance with the LbCS and as per the procedure outlined in the preceding chapters, we now construct attractive and repulsive potential field functions required by the car-like robots and the swarm of boids to successfully avoid each other.

**Assumption 5.3.1.** For all the target attraction and obstacle avoidance functions that will follow, we have $k \in \{1, 2, \ldots, m\}$, $n \in \mathbb{N}$, for $A_k$ and $m \in \{0, 1, \ldots, n\}$, for the $m$th articulated body of $A_k$. 
5.3.1 Details of the Vehicular Agents

Target of the Vehicle

To initiate movement, we propose to have a target for each of the car-like robot. Therefore, for $A_k$, we define a target as follows:

**Definition 5.3.1.** The designated target for the car-like robot of $A_k$ is a disk with center $(p_{k1}, p_{k2})$ and radius $r_{vk}$. That is, it is the set

$$T_k = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - p_{k1})^2 + (z_2 - p_{k2})^2 \leq r_{vk}^2\}.$$ 

Now, in the target-attraction component of the Lyapunov-like function, intuitively, we want to have a kind of a yardstick that measures, at time $t \geq 0$, the midpoint position of $A_k$ from its destination $(p_{k1}, p_{k2})$ and the rate at which it approaches or moves away from $(p_{k1}, p_{k2})$. A choice of probable target attractive functions that could accomplish this, on suppressing $t$, is

$$V_k(x) = \frac{1}{2} \left[ (xv_k - p_{k1})^2 + (yv_k - p_{k2})^2 + v_k^2 + \omega_k^2 \right].$$

(5.5)

While the function is a measure of the distance between $A_k$ and the target $V_k$, it can also be treated as a measure of its convergence with the inclusion of the velocity components [114]. At the target center $(p_{k1}, p_{k2})$, obviously we want the velocities, $v_k$ and $\omega_k$, for $(k = 1, \cdots, m)$, to be zero to ensure that the car-like robot stops and takes on a final orientation. If we assume that this final orientation takes on the values

$$\theta_1 = \theta^f_1, \quad \theta_2 = \theta^f_2, \quad \cdots, \quad \theta_n = \theta^f_n,$$

then the point

$$x^* := (p_{k1}, p_{k2}, \theta^f_1, \theta^f_1, \cdots, \theta^f_n, 0, 0) \in \mathbb{R}^{5x_n}$$

becomes an equilibrium point of the system (5.1). The function $V_k(x)$ will be a component of a total potential function that will be generated by a Lyapunov function $L(x)$ for system (5.1). The global minimum of this Lyapunov-like function should correspond to $x^*$, that is, $L(x^*) = 0$ which implies that $x^*$ is an equilibrium point of system (5.1). The significance of the attractive potential function $V_k(x)$ is to ensure that system trajectories start and remain close to $x^*$, noting that $V_k(x^*) = 0$. In other words, if the car-like robot ever converges to its target, then it remains there at all times.

As an example, Figure 5.2 illustrates the attractive potential fields and the corresponding contour plot generated by equation (5.5) over the workspace $0 < z_1 < 50$ and $0 < z_2 < 25$ for a car-like mobile robot. For 3D visualization and the associated contour plot, the velocity and angular components of the car-like mobile robot have been treated as constants. The disk-shaped target is fixed at $(p_1, p_2) = (25, 12.5)$ with a radius of $r_{vk} = 0.2$. 
Figure 5.2: Attractive potential fields and the corresponding contour plot generated using the target attractive function (see equation (5.5)). The dimensions of the car-like mobile robots are given in Table 5.1, while $v_1 = 2$, $\omega_1 = \pi/360$. For this case, we are only considering one car-like mobile robot.

**Convergence of the Vehicle (Car-like robot)**

We need to guarantee the convergence of the car-like robot to its prescribed target and ensure that the nonlinear controllers vanish at the target configuration [114]. We adopt a new attractive function whose role is purely mathematical, and hence auxiliary [114]. This function will be multiplied to each of the obstacle avoidance functions. This strategy implicitly guarantees that the goal configuration is a *global minimum* of the total potential. This new inclusion also takes care of other associated problems, in particular, goal non-reachable with obstacles nearby (GNRON) [43]. Thus an appropriate auxiliary function is defined as follows:

$$G_k(x) = \frac{1}{2} \left[ (x_{vk} - p_{k1})^2 + (y_{vk} - p_{k2})^2 + (\theta_{vk} - p_{k3})^2 \right] \geq 0. \tag{5.6}$$

**Kinodynamic Constraints**

The kinodynamic planning problem involves synthesizing a robot's motion subject to kinematic constraints, such as any fixed or moving obstacle in the workspace and dynamic constraints, such as modulus bound on velocity. While the nonholonomy of the robotic system, that is, in this case, car-like mobile robot, is reflected in the dynamic model governed by (5.1), the obstacles are either fixed or artificial. The fixed obstacles are the four boundaries of the rectangular workspace, and the stationary solid objects fixed within the workspace. On the other hand, the artificial or ghost obstacles are created to satisfy the dynamic constraints of the system.
Workspace: Boundary Limitations

We consider a simple planar workspace, adopted from [114], so that the car-like robot is confined to the rectangular region at all time $t \geq 0$. Then the following definition is delivered:

**Definition 5.3.2.** The workspace is a fixed, closed and bounded rectangular region, defined, for some constants $b_1$ and $b_2$, as

$$WS = \{(z_1, z_2) \in \mathbb{R}^2 : 0 \leq z_1 \leq b_1, 0 \leq z_2 \leq b_2\},$$

where the boundaries of this region are:

(a) Left Boundary: $B_1 = \{(z_1, z_2) \in \mathbb{R}^2 : z_1 = 0\}$;
(b) Lower Boundary: $B_2 = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 = 0\}$;
(c) Right Boundary: $B_3 = \{(z_1, z_2) \in \mathbb{R}^2 : z_1 = b_1 > 0\}$;
(d) Upper Boundary: $B_4 = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 = b_2 > 0\}$.

In our Lyapunov-based control scheme, these boundaries are considered as fixed obstacles, which have to be avoided by each articulated body at every time $t \geq 0$ so that the robot is confined within the workspace. Accordingly, for their avoidance we construct the following obstacle avoidance functions for the avoidance of the left, lower, right and upper boundaries, respectively, as follows:

$$WV_{k1} = x_{v_k} - r_{v_k},$$
$$WV_{k2} = y_{v_k} - r_{v_k},$$
$$WV_{k3} = b_1 - (x_{v_k} - r_{v_k}),$$
$$WV_{k4} = b_2 - (y_{v_k} - r_{v_k}).$$

(5.7)

Each of these is positive within the rectangle. That is, $WV_{k1}, WV_{k3} > 0$ for all $x_{v_k} \in (r_{v_k}, b_1 - r_{v_k})$ and $WV_{k2}, WV_{k4} > 0$ for all $y_{v_k} \in (r_{v_k}, b_2 - r_{v_k})$.

Now, let us, for the moment, consider, for some parameters $\tau_{ks} > 0$, $k = 1, \ldots, m$ and $s = 1, \ldots, 4$, the effect of the ratios

$$\frac{\tau_{k1}}{WV_{k1}}, \frac{\tau_{k2}}{WV_{k2}}, \frac{\tau_{k3}}{WV_{k3}} \text{ and } \frac{\tau_{k4}}{WV_{k4}}.$$ 

Now, if $A_k$, that is the car-like mobile robot approaches a border of $WS$, then one of the ratios will increase. If $A_k$ moves away from a border, a ratio will decrease. Assume next that the ratios are added appropriately to the total potential field, the Lyapunov function, $L(x)$.
Swarm Navigation in a Dynamic Environment

(that was alluded to in Subsection 5.3.1 and will be proposed in Section 5.4) for system 5.1 as repulsive potential functions, that establishes the stability of system (5.1).

With respect to time \( t \geq 0 \), we have that \( \frac{dL}{dt} \leq 0 \) along a trajectory of (5.1), and \( L \) is a positive definite function, \( L \) cannot increase in \( t \). Thus any change in the value of the ratios could only correspond to either an increase or decrease in \( |\frac{dL}{dt}| \). Analogously, \( |\frac{dL}{dt}| \) is the rate of dissipation of energy from the system in absolute value. If a border is approached, then one of the ratios gets larger. Thus, the rate of energy dissipation, in absolute value, gets larger. This, in turn, instigates an increase in the activity of the system. This increased activity could only be directed towards the equilibrium point, away from the border. In other words, we cannot have a situation where \( WV_{ks} = 0 \), \( k = 1, \ldots, n \), \( s = 1, \ldots, 4 \).

Henceforth, all the obstacle avoidance functions will be appropriately coupled with tuning parameters, in accordance with the Lyapunov-based control scheme. As such, the essence of obstacle avoidance capability in the LbCS lies, therefore, in the creation of obstacle avoidance functions and the repulsive potential functions that will induce an increase or decrease in the instantaneous rate of change of the Lyapunov function.

As an illustration, in Figure 5.3 we show the potential field generated by the function

\[
H(x) + \sum_{s=1}^{16} \frac{\tau_{ks}}{WV_{ks}(x)},
\]

(5.8)

to guide a car-like mobile robot to the target placed at \((25, 12.5)\), the minimum of the field, whilst avoiding the high potential walls that are the workspace boundaries.

Modulus Bound on Velocities

From a practical viewpoint, the translational speed and the steering angle of a car-like system are limited. If \( v_{\text{max}} > 0 \) is the maximum speed, and \( \phi_{\text{max}} \) is the maximum steering angle satisfying \( 0 < \phi_{\text{max}} < \frac{\pi}{2} \) then, as shown in [115], the additional constraints imposed on the translational and the rotational velocities are:

(i) \( |v_k| < v_{\text{max}} \), where \( v_{\text{max}} \) is the maximal achievable speed of the mobile robot;

(ii) \( |\omega_k| \leq \frac{|v_k|}{\rho_{\text{min}}} < \frac{v_{\text{max}}}{\rho_{\text{min}}} \). This is derived from the fact that \( v_k^2 \geq \rho_{\text{min}}^2 \omega_k^2 \) where \( \rho_{\text{min}} \) is known as the minimum turning radius and is given as \( \rho_{\text{min}} = \frac{L}{\tan \phi_{\text{max}}} \). This condition arises due to the boundedness of the steering angle, \( \phi \). That is, \( |\phi| \leq \phi_{\text{max}}, \) where \( \phi_{\text{max}} \) is the maximal steering angle;

The only way these dynamic constraints can be utilised within the LbCS is by constructing artificial obstacles associated to each of the constraints and then avoiding them to accomplish
Figure 5.3: Total potentials and the corresponding contour plot generated from the attractive potentials for target attraction and the repulsive potentials designed for the avoidance of the four borders of the workspace. The target is fixed at \((p_1, p_2) = (25, 12.5)\) with radius \(r_{vk} = 0.2\) for \(k = 1\), while \(\tau_{ks} = 0.01\) for \(s = 1, \ldots, 16\) and \(k = 1\). Also, the dimensions of the car-like mobile robot are given in Table 5.1, while \(v_1 = 2, \omega_1 = \pi/360\).

Using some parameters, say \(\xi_{ku} > 0\), we can therefore form the repulsive potential functions

\[
\sum_{u=1}^{2} \frac{\xi_{ku}}{U_{ku}(x)},
\]

which will provide the appropriate potential fields that will ensure that the car-like mobile robot adheres to within the minimal values.
Moving Obstacles

From a practical viewpoint, satisfactory avoidance of moving obstacles is almost *de rigueur* for mobile robots and control algorithms must be designed that generate feasible trajectories based upon real-time perceptual information. In literature such algorithms are either on-line [54] or off-line [115, 117], the choice depending on the way the information is received and processed by the multi-agents. The on-line algorithms are mostly reactive and localized in nature whilst off-line schemes are generally favored in *a priori* known environments. In this chapter, we will design an off-line avoidance scheme for a group of car-like robots to avoid two different categories of dynamic obstacles.

The moving obstacles of this research paper are as follows:

1. Car-like robots
2. Leader-less swarm

The car-like mobile robots is a rear wheel driven vehicle connected with an axle and with front wheel steering. The leader-less swarm are basically disk shaped with a radius $rb_i$. In here, we model the car-like robots separate from the swarm. We will first discuss the inter-individual collision avoidance for the car-like mobile robots and the leader-less swarm inter-individual avoidance will be described later on.

Inter-individual Collision Avoidance for the Car-like Mobile Robots

The control algorithms must generate feasible trajectories based upon real-time perceptual information. A moving car-like mobile robot itself becomes a moving obstacle for all the other car-like mobile robots in the workspace. First the following definition is made:

**Definition 5.3.3.** The $l$th moving car-like mobile robot is a disk with center $xv_l, yv_l$ and radius $rv_k$. Precisely, the $l$th moving robot is the set

$$A_l = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - xv_l)^2 + (z_2 - yv_l)^2 \leq rv_k\}$$

Also it is necessary to make the following assumption:

**Assumption 5.3.2.** Due to the deterministic nature of our kinodynamic system, there is a prior knowledge of the directions of motion and the instantaneous velocities of the car-like robots available to the system.

For car $A_k$ to avoid car $A_l$, we design repulsive potential field functions with the associated obstacle avoidance function of the form

$$M_{kl}(x) = \frac{1}{2} \left[(xv_k - xv_l)^2 + (yv_k - yv_l)^2 - (rv_k + rv_l)^2\right],$$

for $k, l = 1, \ldots, n, l \neq k$. The function is an Euclidean measure of the distance between $A_k$ and $A_l$, and will appear in the denominator of an appropriate term in the candidate Lyapunov-like function to be proposed.
5.3.2 Details of the Leader-less Swarm

In this subsection, we will adopt the swarm model that was utilised in Chapters 3 and 4. For the attraction of the swarm to the centroid and for the inter-individual avoidance of the swarm, the functions are:

Attraction to the Centroid

We adopt the inter-individual collision avoidance, equation (3.4) that was designed in Chapter 3. To ensure that the individuals of the swarm are attracted towards each other and also form a cohesive group by having a measurement of the distance from the $i$th individual to the swarm centroid, we use the following attraction function:

$$R_i(x) = \frac{1}{2} \left[ \left( x_{bi} - \frac{1}{n} \sum_{j=1}^{n} x_{bj} \right)^2 + \left( y_{bi} - \frac{1}{n} \sum_{j=1}^{n} y_{bj} \right)^2 \right], \quad (5.12)$$

It will be part of the Lyapunov-like function for system (5.4). The role is to ensure that the $i$th individual is attracted to the swarm centroid.

Avoidance of the Boundaries of the Workspace

This subsection adopts the planar workspace $WS$ designed in (5.3.1). For the avoidance of the left, upper, right and lower boundaries, the following functions are utilized, respectively:

$$WB_{i1} = x_{bi} - r_{bi},$$
$$WB_{i2} = y_{bi} - r_{bi},$$
$$WB_{i3} = b_1 - (x_{bi} - r_{bi}),$$
$$WB_{i4} = b_2 - (y_{bi} - r_{bi}), \quad (5.13)$$

where $WS := \{(z_1, z_2) \in \mathbb{R}^2 : 0 \leq z_1 \leq b_1, 0 \leq z_2 \leq b_2\}$ and noting that they are all positive within the WS.

Inter-individual Collision Avoidance

We adopt the inter-individual collision avoidance, equation (3.5) that was designed in Chapter 3. For the boids to avoid each other, we design repulsive function of the form

$$Q_{ij}(x) = \frac{1}{2} \left[ (x_{bi} - x_{bj})^2 + (y_{bi} - y_{bj})^2 - (r_{bi} + r_{bj})^2 \right], \quad (5.14)$$

for $i, j = 1, ..., n, j \neq i$. The function is an Euclidean measure of the distance between the individual boids, and will appear in the denominator of an appropriate term in the candidate Lyapunov-like function to be proposed.
Avoidance of Vehicular Agents by the Boids

In essence, effective avoidance of moving obstacles is essential for mobile robots. Hence, avoidance of the moving swarms is another addition to the multi-tasking problem in this chapter. Here, the car-like mobile robots becomes the moving obstacles for the swarm of boids in the workspace. There are \( m \in \mathbb{N} \) vehicular agents. This is one-way collision avoidance whereby the swarm of boids avoid the car-like mobile robots. This is achieved via tuning parameters via appropriate optimisation techniques. For the boids to avoid the vehicular agents, we design repulsive potential field functions of the form

\[
S_{ik} = \frac{1}{2} \left[ (xb_i - xv_k)^2 + (yb_i - xv_k)^2 - (rb_i + rv_k)^2 \right],
\]

where \( k = 1, 2, \ldots, m \). This will form a part of the tentative Lyapunov function and will appear in the denominator with an appropriate cohesion parameter.

5.4 Design of Nonlinear Controllers

This section will represent a Lyapunov-like function candidate and the nonlinear control laws for systems (5.1) and (5.4) will be designed using the LbCS. In parallel, we will consider the stability analysis pertaining to the dynamic system.

5.4.1 Lyapunov-like Function

As per the LbCS, we combine all the attractive and repulsive potential field functions, and introducing tuning parameters (or control parameters), \( \gamma_i > 0, \eta_{is} > 0, \beta_{ij} > 0, \sigma_{ik} > 0, \tau_{ks} > 0, \varphi_{kl} > 0, \) and \( \xi_{ku} > 0 \) for \( i, j, k, l, m, s, u \in \mathbb{N} \), we define a Lyapunov function candidate for systems (5.1) and (5.4) as

\[
L(x) = \sum_{i=1}^{n} \left[ \gamma_i R_i(x) + R_i(x) \left( \sum_{s=1}^{4} \frac{\eta_{is}}{WB_{is}(x)} + \sum_{j=1}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} + \sum_{k=1}^{m} \frac{\sigma_{ik}}{S_{ik}(x)} \right) \right] + \sum_{k=1}^{m} \left[ V_k(x) + G_k(x) \left( \sum_{s=1}^{4} \frac{\tau_{ks}}{WV_{ks}(x)} + \sum_{l=1}^{m} \frac{\varphi_{kl}}{M_{kl}(x)} + \sum_{u=1}^{2} \frac{\xi_{ku}}{U_{ku}(x)} \right) \right],
\]

\( (5.16) \)

5.4.2 Controller Design

To extract the control laws for the kinodynamic system, we differentiate the various components of \( L(x) \) separately with respect to \( t \) along a solution of systems (5.1) and (5.4), carry out the necessary substitutions and upon suppressing \( x \), we have the following for the swarm of boids and the vehicular agents:
Swarm of boids

Upon suppressing $x$ and for $i = 1, 2, \ldots, n$, we have

$$\begin{align*}
Lx_i &= \left( \gamma_i + \sum_{j=1}^{n} \frac{\beta_{ij}}{Q_{ij}} \right) \left( xb_i - \frac{1}{n} \sum_{j=1}^{n} xb_j \right) + R_i \left( \frac{\eta_3}{WB_{i3}^2} - \frac{\eta_1}{WB_{i1}^2} \right) - 2R_i \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij}}{Q_{ij}^2} (xb_i - xb_j) \\
&\quad - R_i \sum_{k=1}^{m} \frac{\sigma_{ik}}{S_{ik}^2} (xb_i - xv_k), \\
Ly_i &= \left( \gamma_i + \sum_{j=1}^{n} \frac{\beta_{ij}}{Q_{ij}} \right) \left( yb_i - \frac{1}{n} \sum_{j=1}^{n} yb_j \right) + R_i \left( \frac{\eta_4}{WB_{i4}^2} - \frac{\eta_2}{WB_{i2}^2} \right) - 2R_i \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij}}{Q_{ij}^2} (yb_i - yb_j) \\
&\quad - R_i \sum_{k=1}^{m} \frac{\sigma_{ik}}{S_{ik}^2} (yb_i - yv_k).
\end{align*}$$

Next, given the convergence parameters $\alpha_{i1}, \alpha_{i2} > 0$, the nonlinear velocity controllers for the swarm of boids is:

$$\begin{align*}
v_{bi} &= -\alpha_{i1} Lx_i, \\
\omega_{bi} &= -\alpha_{i2} Ly_i,
\end{align*}$$

(5.17)

where $i = 1, 2, \ldots, n$.

Vehicular Agents

Upon suppressing $x$ and for $k = 1, 2, \ldots, m$, we have

$$\begin{align*}
f_{k1} &= \left( 1 + \sum_{s=1}^{4} \frac{\tau_{ks}}{WV_{ks}(x)} + \sum_{l=1}^{m} \frac{\varphi_{kl}}{M_{kl}(x)} + \sum_{u=1}^{2} \frac{\xi_{ku}}{U_{ku}(x)} \right) (xv_k - p_{k1}) \\
&\quad + G_k \left( \frac{\tau_{k3}}{WV_{k3}^2} - \frac{\tau_{k1}}{WV_{k1}^2} \right) - G_k \sum_{l=1}^{m} \frac{\varphi_{kl}}{M_{kl}^2} (xv_k - xv_l) + G_k \sum_{l=1}^{m} \frac{\varphi_{lk}}{M_{lk}^2} (xv_l - xv_k),
\end{align*}$$
\begin{align*}
\tau_{k2} &= \left(1 + \sum_{s=1}^{4} \frac{\tau_{ks}}{WV_{ks}(x)} + \sum_{l=1 \atop l \neq k}^{m} \frac{\varphi_{kl}}{M_{kl}(x)} + \sum_{u=1}^{2} \frac{\xi_{ku}}{U_{ku}(x)}\right) (yv_k - p_k) \\
&\quad + G_k \left(\frac{\tau_{k4}}{WV_{k4}^2} - \frac{\tau_{k2}}{WV_{k2}^2}\right) - G_k \sum_{l=1 \atop l \neq k}^{m} \frac{\varphi_{kl}}{M_{kl}^2} (yv_k - yv_l) + G_l \sum_{l=1 \atop l \neq k}^{m} \frac{\varphi_{lk}}{M_{lk}^2} (yv_l - yv_k),
\end{align*}

\begin{align*}
g_{k1} &= 1 + G_k \frac{\xi_{k1}}{U_{k1}^2}, \\
g_{k2} &= 1 + G_k \frac{\xi_{k2}}{U_{k2}^2}.
\end{align*}

Next, given convergence parameters \( \delta_{k1}, \delta_{k2} > 0 \), the translational and rotational speeds are given the following forms:

\begin{align*}
-\delta_{k1} \times v_k &= \frac{(f_{k1}(x) \cos \theta_k + f_{k2}(x) \sin \theta_k) + g_{k1}(x)u_{k1}}{g_{k1}}, \\
-\delta_{k2} \times \omega_k &= \frac{L}{2} \left( f_{k2}(x) \cos \theta_k - f_{k1}(x) \sin \theta_k \right) + g_{k2}(x)u_{k2},
\end{align*}

where \( k = 1, 2, \ldots, m \) and \( L \) is the length of the \( k \)-th car.

Hence, along a trajectory of system (5.1)

\[ \dot{L}(x) = -\sum_{k=1}^{m} \left( \delta_{k1} v_k^2 + \delta_{k2} \omega_k^2 \right) \leq 0 \]

provided that the state feedback nonlinear navigation laws governing the \( k \)-th car are of the form

\begin{align*}
u_{k1} &= -(\delta_{k1} v_k + f_{k1} \cos \theta_k + f_{k2} \sin \theta_k) / g_{k1}, \\
u_{k2} &= -\left[ \delta_{k2} \omega_k + \frac{L}{2} (f_{k2} \cos \theta_k - f_{k1} \sin \theta_k) \right] / g_{k2}.
\end{align*}

Note that \( \dot{L}(x) \leq 0 \) for all \( x \in D(L) \), and \( L(x)(x^*) = 0 \).

### 5.5 Stability Analysis

**Theorem 5.5.1.** System (5.1) and (5.4) is uniformly practically stable.

**Proof.** The proof is very similar to section 3.5.
5.6 Simulation 1

This section demonstrates the effectiveness of the nonlinear control laws with simulation results from virtual scenarios. Our simple setup is where car-like mobile robots have to navigate from an initial to a final configuration, whilst avoiding moving obstacles en route their target. The moving obstacles are the swarm of boids in each of the cases. The stability results obtained from the Lyapunov-like function will be verified numerically.

5.6.1 Scenario 1

In this scenario, the car-like mobile robots move from an initial configuration to the target position whilst avoiding each other and the swarm of boids on their way to their target. This scenario could be modeled as a herd of cows or elephants following a car from one destination to another. We have strategically placed the car-like mobile robots on the route of the emergent behavior exhibited by the swarm of boids to mimic the situation wherein the car-like robots have in a workspace or road with cows or elephants on it.

The nonlinear controllers $u_{k1}$ and $u_{k2}$, for $k = 1, 2$ were simulated to generate a feasible robot trajectory. With the initial conditions described in Table 5.1, the control laws ensured a nice convergence of the system state to the final state, whilst satisfying all underlying constraints.

Figure 5.5 shows the profile of the Lyapunov-like function and Figure 5.6 shows the time derivative of the Lyapunov-like function along the system trajectories. Figures 5.7 and 5.8, and 5.9 shows the velocity components of the third boid in the swarm. One can clearly notice the convergence of the variables showing the cohesiveness of the swarm. Similar convergent trends were exhibited by other boids in the swarm. The rest of the scenarios exhibited almost similar convergent behaviors.
Table 5.1: Scenario 1. Numerical values of initial state, constraints, and control and convergence parameters. There are 15 boids and 2 car-like mobile robots.

<table>
<thead>
<tr>
<th></th>
<th>Initial state of the car-like robots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular position</td>
<td>$(xv_1, yv_1) = (40, 40)$</td>
</tr>
<tr>
<td></td>
<td>$(xv_2, yv_2) = (60, 460)$</td>
</tr>
<tr>
<td>Angular positions</td>
<td>$\theta_1 = \pi/4$, $\theta_2 = -\pi/4$</td>
</tr>
<tr>
<td>Translational velocities</td>
<td>$v_1 = v_2 = 3$</td>
</tr>
<tr>
<td>Rotational velocities</td>
<td>$\omega_1 = \omega_2 = 0.3$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimension of robot</td>
<td>$L = 20$, $\ell = 10$</td>
</tr>
<tr>
<td>Targets</td>
<td>$(p_{11}, p_{12}) = (460, 460)$ and radius $rt_1 = 10$</td>
</tr>
<tr>
<td></td>
<td>$(p_{21}, p_{22}) = (460, 40)$ and radius $rt_2 = 10$</td>
</tr>
<tr>
<td>Max. translational velocity</td>
<td>$v_{\text{max}} = 10$</td>
</tr>
<tr>
<td>Max. steering angle</td>
<td>$\phi_{\text{max}} = 7\pi/18$</td>
</tr>
<tr>
<td>Clearance parameters</td>
<td>$\epsilon_1 = 0.1$, $\epsilon_2 = 0.2$</td>
</tr>
<tr>
<td>Workspace boundaries</td>
<td>$b_1 = b_2 = 500$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Parameters for the boids</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary limitations</td>
<td>$\eta_{is} = 0.0001$ for $i = 1, \ldots, 15$ and $s = 1, \ldots, 4$</td>
</tr>
<tr>
<td>Cohesion parameter</td>
<td>$\gamma_i = 20$, for $i = 1, \ldots, 15$</td>
</tr>
<tr>
<td>Coupling/inter-individual obstacle avoidance</td>
<td>$\beta_{ij} = \text{random}[200, 500]$, for $i, j = 1, \ldots, 15, i \neq j$</td>
</tr>
<tr>
<td>Avoidance of agents by boids</td>
<td>$\sigma_{ik} = 1$, for $i = 1, \ldots, 15$ and $k = 1, 2$</td>
</tr>
<tr>
<td>Convergence</td>
<td>$\alpha_{i1} = \alpha_{i2} = 0.001$, for $i = 1, \ldots, 15$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Parameters for the car-like robots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inter-individual avoidance</td>
<td>$\varphi_{kl} = 1$, for $k, l = 1, 2$, $k \neq l$</td>
</tr>
<tr>
<td>Modulus bound on velocity</td>
<td>$\xi_{ku} = 0.00001$, for $k = 1, 2$, and $u = 1, 2$</td>
</tr>
<tr>
<td>Convergence</td>
<td>$\delta_{k1} = \delta_{k2} = 5000$, for $k = 1, 2$</td>
</tr>
</tbody>
</table>
Figure 5.4: There are $n = 15$ boids (shown in red), each with bin size 5, randomly positioned at the initial time $t = 0$. The horizontal and vertical axes give the coordinates $x_i(t)$ and $y_i(t)$ as $z_1(t)$ and $z_2(t)$, respectively, for each individual $i$ at time $t \geq 0$. The grey lines show the trajectories and of the individuals. The path of the centroid is given by the green line. The car-like agents are shown in blue color.
Figure 5.5: Scenario 1. Evolution of the Lyapunov-like function $L_i(x)$ of the swarm of boids and the car-like mobile robots.

Figure 5.6: Scenario 1. The evolution of the time derivative of the Lyapunov-like function $\dot{L}_i(x)$.

Figure 5.7: Scenario 1. The velocity $v_3$ of the swarm.

Figure 5.8: Scenario 1. The velocity $w_3$ of the swarm.
Figure 5.9: Scenario 1. The velocity $v_3$ and $w_3$ of the swarm when seen together.

Figure 5.10: Scenario 1. Progression of the velocity components of the car $A_1$. The blue line shows the velocity $v_1$ and the purple line shows the velocity $w_1$.

Figure 5.11: Scenario 1. Evolution of the acceleration controllers of $A_1$. 
5.6.2 Scenario 2

Scenario 2 mimics a split-rejoin maneuver of the car-like mobile robots, wherein the car-like mobile agents avoid each other and the swarm of boids avoid the car in a unique emergent behavior before the car-like mobile robots reach its designated target. Table 5.2 provides all the values of the initial conditions, constraints and different parameters utilized in the simulation, if different from the previous scenario.

In this scenario, we have strategically used the randomised method which placed the individual boids randomly within the workspace. The swarm then forms an emergent multi circular behavior that avoids the car-like mobile robots and follows one of it to its designated target.

Table 5.2: Scenario 2. Numerical values of initial state, constraints, and control and convergence parameters. There are 15 boids and 3 car-like mobile robots.

<table>
<thead>
<tr>
<th>Initial State of the car-like robots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular position</td>
</tr>
<tr>
<td>$(x_{v1}, y_{v1}) = (40, 40)$</td>
</tr>
<tr>
<td>$(x_{v2}, y_{v2}) = (40, 460)$</td>
</tr>
<tr>
<td>$(x_{v3}, y_{v3}) = (40, 250)$</td>
</tr>
<tr>
<td>Angular positions</td>
</tr>
<tr>
<td>$\theta_1 = \pi/4, \theta_2 = -\pi/4$ and $\theta_3 = \pi$</td>
</tr>
<tr>
<td>Translational velocities</td>
</tr>
<tr>
<td>$v_1 = v_2 = v_3 = 3$</td>
</tr>
<tr>
<td>Rotational velocities</td>
</tr>
<tr>
<td>$\omega_1 = \omega_2 = \omega_3 = 0.3$</td>
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<td>Dimension of robot</td>
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</tr>
<tr>
<td>Targets</td>
</tr>
<tr>
<td>$(p_{11}, p_{12}) = (460, 460)$ and radius $rt_1 = 10$</td>
</tr>
<tr>
<td>$(p_{21}, p_{22}) = (460, 40)$ and radius $rt_2 = 10$</td>
</tr>
<tr>
<td>$(p_{31}, p_{32}) = (460, 250)$ and radius $rt_3 = 10$</td>
</tr>
<tr>
<td>Max. translational velocity</td>
</tr>
<tr>
<td>$v_{max} = 10$</td>
</tr>
<tr>
<td>Max. steering angle</td>
</tr>
<tr>
<td>$\phi_{max} = 7\pi/18$</td>
</tr>
<tr>
<td>Clearance parameters</td>
</tr>
<tr>
<td>$\epsilon_1 = 0.1$, $\epsilon_2 = 0.2$</td>
</tr>
<tr>
<td>Workspace boundaries</td>
</tr>
<tr>
<td>$b_1 = b_2 = 500$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters for the boids</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary limitations</td>
</tr>
<tr>
<td>$\eta_{is} = 0.0001$ for $i = 1, \ldots, 15$</td>
</tr>
<tr>
<td>and $s = 1, \ldots, 4$</td>
</tr>
<tr>
<td>Cohesion parameter</td>
</tr>
<tr>
<td>$\gamma_i = 25$, for $i = 1, \ldots, 15$</td>
</tr>
<tr>
<td>Coupling/inter-individual obstacle avoidance</td>
</tr>
<tr>
<td>$\beta_{ij} =$random$[100, 500]$,</td>
</tr>
<tr>
<td>for $i, j = 1, \ldots, 15$, $i \neq j$</td>
</tr>
<tr>
<td>Avoidance of agents by boids</td>
</tr>
<tr>
<td>$\sigma_{ik} = 1$, for $i = 1, \ldots, 15$ and $k = 1, 2, 3$</td>
</tr>
<tr>
<td>Convergence</td>
</tr>
<tr>
<td>$\alpha_{i1} = \alpha_{i2} = 0.01$, for $i = 1, \ldots, 15$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters for the car-like robots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inter-individual avoidance</td>
</tr>
<tr>
<td>$\varphi_{kl} = 5$, for $k, l = 1, 2, 3$, $k \neq l$</td>
</tr>
<tr>
<td>Modulus bound on velocity</td>
</tr>
<tr>
<td>$\xi_{ku} = 0.0001$, for $k, u = 1, 2, 3$</td>
</tr>
<tr>
<td>Convergence</td>
</tr>
<tr>
<td>$\delta_{k1} = \delta_{k2} = 5000$, for $i = 1, 2, 3$</td>
</tr>
</tbody>
</table>
(a) The initial position of the car-like mobile robots and the swarm of boids. ($t = 0$).

(b) The car-like mobile robots avoiding the swarm of boids at $t = 300$ units.

(c) The car-like mobile robots avoiding the swarm of boids at $t = 400$ units.

(d) The final posture of the car-like mobile robots and the position of the swarm at $t = 500$ units.

Figure 5.12: There are $n = 15$ boids (shown in red), each with bin size 5, randomly positioned at the initial time $t = 0$. The horizontal and vertical axes give the coordinates $x_i(t)$ and $y_i(t)$ as $z_1(t)$ and $z_2(t)$, respectively, for each individual $i$ at time $t \geq 0$. The grey lines show the trajectories and of the individuals. The path of the centroid is given by the green line. The car-like agents are shown in blue color.
5.6.3 Scenario 3

Scenario 3 mimics a circular behavior exhibited by the swarm of boids, wherein the car-like mobile agents steer in the workspace to reach its designated target. Initially, the swarm are randomly placed within the workspace and the car-like robots are given its initial conditions. The swarm of boids then come to their center and start exhibiting a circular behavior. The behavior that is exhibited varies from animals to animals. The swarm circle and in the midst, the car-like mobile robots considers the swarm as obstacles and avoids it reaching its designated target. Table 5.3 provides all the values of the initial conditions, constraints and different parameters utilized in the simulation, if different from the previous scenario.

<table>
<thead>
<tr>
<th>Parameters for the boids</th>
<th>Boundary limitations</th>
<th>$\eta_{is} = 0.0005$ for $i = 1,\ldots,15$ and $s = 1,\ldots,4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cohesion parameter</td>
<td>$\gamma_i = 60$, for $i = 1,\ldots,15$</td>
<td></td>
</tr>
<tr>
<td>Coupling/inter-individual obstacle avoidance</td>
<td>$\beta_{ij} =$random$[200, 600]$, for $i, j = 1,\ldots,15$, $i \neq j$</td>
<td></td>
</tr>
<tr>
<td>Avoidance of agents by boids</td>
<td>$\sigma_{ik} = 2$, for $i = 1,\ldots,15$ and $k = 1,2,3$</td>
<td></td>
</tr>
<tr>
<td>Convergence</td>
<td>$\alpha_{i1} = \alpha_{i2} = 0.003$, for $i = 1,\ldots,15$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters for the car-like robots</th>
<th>Inter-individual avoidance</th>
<th>$\varphi_{kl} = 3.5$, for $k,l = 1,2,3$, $i \neq j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulus bound on velocity</td>
<td>$\xi_{ku} = 0.00005$, for $k,u = 1,2,3$</td>
<td></td>
</tr>
<tr>
<td>Convergence</td>
<td>$\delta_{k1} = \delta_{k2} = 5400$, for $k = 1,2,3$</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.13: There are \( n = 15 \) boids (shown in red), each with bin size 5, randomly positioned at the initial time \( t = 0 \). The horizontal and vertical axes give the coordinates \( x_i(t) \) and \( y_i(t) \) as \( z_1(t) \) and \( z_2(t) \), respectively, for each individual \( i \) at time \( t \geq 0 \). The grey lines show the trajectories and of the individuals. The path of the centroid is given by the green line. The car-like agents are shown in blue color.
5.7 Swarming in the Presence of Fixed Obstacles

In this section, we introduce fixed disk-shaped obstacles into the workspace. In nature, there is presence of both fixed and moving obstacles which needs to be avoided by any moving body. Since the objects can be of any shape (regular or irregular), a plan will need to be devised to ensure that the entire body of an object is enveloped within the defined obstacle space. We shall use the concept of stationary obstacles that was defined in Chapter 4.

5.7.1 Obstacle Avoidance by the Swarm of Boids

Let us fix $z \in \mathbb{N}$ disk-shaped obstacles within the workspace, WS. The $r$th disk-shaped obstacle is defined as a circular disk with center given as $(o_{r1}, o_{r2})$ and with radius $ro_r$. For avoidance by the $i$th body of the swarm, we consider the obstacle avoidance function

$$W_{ir} = \frac{1}{2} \left[ (xb_i - o_{r1})^2 + (yb_i - o_{r2})^2 - (rb_i + ro_r)^2 \right], \quad (5.19)$$

where $r = 1, 2, \ldots, z$.

5.7.2 Obstacle Avoidance by the Vehicular Agents

Let us fix $z \in \mathbb{N}$ disk-shaped obstacles within the workspace, WS. The $r$th disk-shaped obstacle is defined as a circular disk with center given as $(o_{r1}, o_{r2})$ and with radius $ro_r$. For avoidance by the $k$th body of $A_i$, we consider the obstacle avoidance function

$$F_{kr} = \frac{1}{2} \left[ (xv_k - o_{r1})^2 + (yv_k - o_{r2})^2 - (rv_k + ro_r)^2 \right], \quad (5.20)$$

where $r = 1, 2, \ldots, z$.

5.8 Design of Nonlinear Controllers and Stability Analysis

Again, the principle objective is to design artificial potential functions (APFs) from the Lyapunov-based control scheme (LbCS), and accordingly derive the decentralized acceleration controls $u_{k1}$ and $u_{k2}$ such that the car-like mobile robots will navigate safely in the workspace, reach a neighborhood of its target and be aligned to a predefined final posture.

5.8.1 Lyapunov Function

As per the LbCS, we include repulsive potential field functions, equations (5.19) and (5.20) into the Lyapunov-like function from the previous section, equation (5.16), and introducing
tuning parameters (or control parameters), $\zeta_{ir} > 0$, and $\lambda_{kr} > 0$ for $i, k, r, \in \mathbb{N}$, we redefine a Lyapunov function candidate for systems (5.1) and (5.4) as

$$
L(x) = \sum_{i=1}^{n} \left( \gamma_i R_i(x) + R_j(x) \left( \sum_{s=1}^{4} \frac{\eta_{is}}{WB_{is}(x)} + \sum_{j=1}^{n} \frac{\beta_{ij}}{Q_{ij}(x)} + \sum_{r=1}^{z} \frac{\zeta_{ir}}{W_{ir}(x)} + \sum_{k=1}^{m} \frac{\sigma_{ik}}{S_{ik}(x)} \right) \right) 
$$

$$
+ \sum_{k=1}^{m} \left( V_k(x) + G_k(x) \left( \sum_{s=1}^{4} \frac{\tau_{ks}}{WV_{ks}(x)} + \sum_{l=1}^{m} \frac{\varphi_{kl}}{M_{kl}(x)} + \sum_{r=1}^{z} \frac{\lambda_{kr}}{F_{kr}(x)} + \sum_{u=1}^{2} \frac{\xi_{ku}}{U_{ku}(x)} \right) \right) 
$$

(5.21)

### 5.8.2 Controller Design

To extract the control laws for the kinodynamic system, we differentiate the various components of $L(x)$ separately with respect to $t$ along a solution of systems (5.1) and (5.4), carry out the necessary substitutions and upon suppressing $x$, we have the following for the swarm of boids and the vehicular agents:

**Swarm of boids**

Upon suppressing $x$ and for $i = 1, 2, \ldots, n$, we have

$$
Lx_i = \left( \gamma_i + \sum_{j=1}^{n} \frac{\beta_{ij}}{Q_{ij}} \right) \left( xb_i - \frac{1}{n} \sum_{j=1}^{n} xb_j \right) + R_i \left( \frac{\eta_{3i}}{WB_{i3}} - \frac{\eta_{1i}}{WB_{i1}} \right) - 2R_i \sum_{j=1}^{n} \frac{\beta_{ij}}{Q_{ij}^2} (xb_i - xb_j)
$$

$$
- R_i \sum_{r=1}^{z} \frac{\zeta_{ir}}{W_{ir}^2} (xb_i - \alpha_{1r}) - R_i \sum_{k=1}^{m} \frac{\sigma_{ik}}{S_{ik}^2} (xb_i - xv_k),
$$

$$
Ly_i = \left( \gamma_i + \sum_{j=1}^{n} \frac{\beta_{ij}}{Q_{ij}} \right) \left( yb_i - \frac{1}{n} \sum_{j=1}^{n} yb_j \right) + R_i \left( \frac{\eta_{4i}}{WB_{i4}} - \frac{\eta_{2i}}{WB_{i2}} \right) - 2R_i \sum_{j=1}^{n} \frac{\beta_{ij}}{Q_{ij}^2} (yb_i - yb_j)
$$

$$
- R_i \sum_{r=1}^{z} \frac{\zeta_{ir}}{W_{ir}^2} (yb_i - \alpha_{2r}) - R_i \sum_{k=1}^{m} \frac{\sigma_{ik}}{S_{ik}^2} (yb_i - yv_k).
$$

Next, given the convergence parameters $\alpha_{1i}, \alpha_{2i} > 0$, the nonlinear velocity controllers for the swarm of boids is:

$$
v_{bi} = -\alpha_{1i} Lx_i,
$$

$$
\omega_{bi} = -\alpha_{2i} Ly_i,
$$

(5.22)

where $i = 1, 2, \ldots, n$. 
Vehicular Agents

Upon suppressing $x$ and for $k = 1, 2, \ldots, m$, we have

\[
\begin{align*}
    f_{k1} &= \left( 1 + \sum_{s=1}^{4} \frac{\tau_{ks}}{WV_{k}(x)} + \sum_{l=1}^{m} \frac{\varphi_{kl}}{M_{kl}(x)} + \sum_{r=1}^{z} \frac{\lambda_{kr}}{F_{kr}(x)} + \sum_{u=1}^{2} \frac{\xi_{ku}}{U_{ku}(x)} \right) (xv_{k} - p_{k1}) \\
    &\quad + G_{k} \left( \frac{\tau_{k3}}{WV_{k}^{2}k_{3}} - \frac{\tau_{k1}}{WV_{k}^{2}k_{1}} \right) - G_{k} \sum_{l=1}^{m} \frac{\varphi_{kl}}{M_{kl}^{2}} (xv_{k} - xv_{l}) + G_{k} \sum_{l=1}^{m} \frac{\varphi_{lk}}{M_{lk}^{2}} (yv_{k} - yv_{l}) \\
    &\quad - G_{k} \sum_{r=1}^{z} \frac{\lambda_{kr}}{F_{kr}^{2}} (xv_{k} - o_{r1}), \\
    f_{k2} &= \left( 1 + \sum_{s=1}^{4} \frac{\tau_{ks}}{WV_{k}(x)} + \sum_{l=1}^{m} \frac{\varphi_{kl}}{M_{kl}(x)} + \sum_{r=1}^{z} \frac{\lambda_{kr}}{F_{kr}(x)} + \sum_{u=1}^{2} \frac{\xi_{ku}}{U_{ku}(x)} \right) (yv_{k} - p_{k2}) \\
    &\quad + G_{k} \left( \frac{\tau_{k4}}{WV_{k}^{2}k_{4}} - \frac{\tau_{k2}}{WV_{k}^{2}k_{2}} \right) - G_{k} \sum_{l=1}^{m} \frac{\varphi_{kl}}{M_{kl}^{2}} (yv_{k} - yv_{l}) + G_{k} \sum_{l=1}^{m} \frac{\varphi_{lk}}{M_{lk}^{2}} (yv_{k} - yv_{l}) \\
    &\quad - G_{k} \sum_{r=1}^{z} \frac{\lambda_{kr}}{F_{kr}^{2}} (yv_{k} - o_{r2}), \\
    g_{k1} &= 1 + G_{k} \frac{\xi_{k1}}{U_{k1}^{2}}, \\
    g_{k2} &= 1 + G_{k} \frac{\xi_{k2}}{U_{k2}^{2}}.
\end{align*}
\]

Next, given convergence parameters $\delta_{k1}, \delta_{k2} > 0$, the translational and rotational speeds are given the following forms:

\[
\begin{align*}
    -\delta_{k1} \times v_{k} &= (f_{k1}(x) \cos \theta_{k} + f_{k2}(x) \sin \theta_{k} + g_{k1}(x)u_{k1}), \\
    -\delta_{k2} \times \omega_{k} &= \frac{L}{2} (f_{k2}(x) \cos \theta_{k} - f_{k1}(x) \sin \theta_{k}) + + g_{k2}(x)u_{k2},
\end{align*}
\]

where $k = 1, 2, \ldots, m$ and $L$ is the length of the $k$th car.

Hence, along a trajectory of system (5.1)

\[
\dot{L}(x) = - \sum_{k=1}^{m} \left( \delta_{k1} v_{k}^{2} + \delta_{k2} \omega_{k}^{2} \right) \leq 0
\]
provided that the state feedback nonlinear control laws governing the $k$th car are of the form

$$u_{k1} = -\left(\delta_{k1} v_k + f_{k1} \cos \theta_k + f_{k2} \sin \theta_k\right)/g_{k1},$$

$$u_{k2} = -\left[\delta_{k2} \omega_k + \frac{\ell_k}{2} (f_{k2} \cos \theta_k - f_{k1} \sin \theta_k)\right]/g_{k2}. \quad (5.23)$$

Note that $\dot{L}(x) \leq 0$ for all $x \in D(L)$, and $\dot{L}(x^*) = 0$.

## 5.9 Stability Analysis

**Theorem 5.9.1.** System (5.1) and (5.4) is uniformly practically stable.

**Proof.** The proof is very similar to section 3.5.

## 5.10 Simulation 2

This section demonstrates the simulation results for the car-like mobile robots navigating in a well defined workspace cluttered with fixed and moving obstacles. The fixed obstacles are randomly placed disks within the workspace and the moving obstacles are the car-like mobile robots avoiding each other and the swarm of boids which exhibit an emergent behavior. We verify numerically the stability results using the Lyapunov-like function.

### 5.10.1 Scenario 1

This scenario considers 2 car-like mobile robots and 10 boids and depicts a virtual scenario whereby a flock of birds move about in a workspace cluttered with obstacles. The corresponding initial and final states, and other essential data are provided in Table 5.4.
Table 5.4: *Scenario 1*. Numerical values of initial state, constraints, and control and convergence parameters.

<table>
<thead>
<tr>
<th><strong>Initial state of the car-like robots</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rectangular positions</strong></td>
<td>((x_{v1}, y_{v1}) = (40, 40))</td>
</tr>
<tr>
<td></td>
<td>((x_{v2}, y_{v2}) = (60, 460))</td>
</tr>
<tr>
<td><strong>Angular positions</strong></td>
<td>(\theta_1 = \pi/4), and (\theta_2 = -\pi/4)</td>
</tr>
<tr>
<td><strong>Translational velocities</strong></td>
<td>(v_1 = v_2 = 3)</td>
</tr>
<tr>
<td><strong>Rotational velocities</strong></td>
<td>(\omega_1 = \omega_2 = 0.3)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Constraints</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dimension of robot</strong></td>
<td>(L = 20), (\ell = 10)</td>
</tr>
<tr>
<td><strong>Targets</strong></td>
<td>((p_{11}, p_{12}) = (460, 460)) and radius (r_{t1} = 10)</td>
</tr>
<tr>
<td></td>
<td>((p_{21}, p_{22}) = (460, 40)) and radius (r_{t2} = 10)</td>
</tr>
<tr>
<td><strong>Max. translational velocity</strong></td>
<td>(v_{\text{max}} = 10)</td>
</tr>
<tr>
<td><strong>Max. steering angle</strong></td>
<td>(\phi_{\text{max}} = 7\pi/18)</td>
</tr>
<tr>
<td><strong>Clearance parameters</strong></td>
<td>(\epsilon_1 = 0.1), (\epsilon_2 = 0.2)</td>
</tr>
<tr>
<td><strong>Workspace boundaries</strong></td>
<td>(b_1 = b_2 = 500)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Parameters for the boids</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Boundary limitations</strong></td>
<td>(\eta_{ks} = \tau_{ks} = 0.0001) for (i = 1, \ldots, 10), (k = 1, 2) and (s = 1, \ldots, 4)</td>
</tr>
<tr>
<td><strong>Cohesion parameter</strong></td>
<td>(\gamma_i = 2), for (i = 1, \ldots, 10)</td>
</tr>
<tr>
<td><strong>Coupling/inter-individual obstacle avoidance</strong></td>
<td>(\beta_{ij} = 5), for (i, j = 1, \ldots, 10), (i \neq j)</td>
</tr>
<tr>
<td><strong>Avoidance of agents by boids</strong></td>
<td>(\sigma_{ik} = 1), for (i = 1, \ldots, 10) and (k = 1, 2)</td>
</tr>
<tr>
<td><strong>Convergence</strong></td>
<td>(\alpha_{i1} = \alpha_{i2} = 0.01), for (i = 1, \ldots, 10)</td>
</tr>
<tr>
<td><strong>Avoidance of fixed obstacles</strong></td>
<td>(\zeta_{ir} = \text{random}[3, 5]), for (i = 1, \ldots, 10), and (k = 1, 2)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Parameters for the car-like robots</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Inter-individual avoidance</strong></td>
<td>(\varphi_{kl} = 2), for (k, l = 1, 2), (k \neq l)</td>
</tr>
<tr>
<td><strong>Modulus bound on velocity</strong></td>
<td>(\xi_{ku} = 0.0001), for (k, u = 1, 2)</td>
</tr>
<tr>
<td><strong>Avoidance of fixed obstacles</strong></td>
<td>(\lambda_{kr} = 1.5), for (k = 1, 2), and (r = 1, \ldots, 5)</td>
</tr>
<tr>
<td><strong>Convergence</strong></td>
<td>(\delta_{k1} = \delta_{k2} = 4500), for (k = 1, 2)</td>
</tr>
</tbody>
</table>
Figure 5.14: There are $n = 10$ boids (shown in red), each with bin size 3, randomly positioned at the initial time $t = 0$. The horizontal and vertical axes give the coordinates $x_i(t)$ and $y_i(t)$ as $z_1(t)$ and $z_2(t)$, respectively, for each individual $i$ at time $t \geq 0$. The grey lines show the trajectories and of the individuals. The path of the centroid is given by the green line. The car-like agents are shown in blue color. There are 5 randomly placed obstacles shown in blue.
Figure 5.15: Scenario 1. Evolution of the Lyapunov function $L_i(x)$ of the swarm of boids and the car-like mobile robots.

Figure 5.16: Scenario 1. The evolution of the time derivative of the Lyapunov function $\dot{L}_i(x)$.

Figure 5.17: Scenario 1. The velocity $v_3$ of the swarm.

Figure 5.18: Scenario 1. The velocity $w_3$ of the swarm.
5.10.2 Scenario 2

In this scenario, a traffic-like situation is mimicked whereby the car-like mobile robots move from an initial configuration to the target position whilst avoiding each other, the disk shaped obstacles and the swarm of boids on their way to their target. We have strategically placed the car-like mobile robots on the route of the emergent behavior exhibited by the swarm of boids to mimic the situation wherein the car-like robots have to avoid heavy traffic.

The nonlinear controllers $u_{k1}$ and $u_{k2}$, for $k = 1, 2, 3$ were simulated to generate a feasible robot trajectory. Noting that the number of boids is 15 and the car-like mobile robots is 3 and assuming the units above been appropriately taken care of, initial conditions pertaining the kinodynamic system (5.2) and other essentials of the situation are provided in Table 5.5.
(a) The initial position of the car-like mobile robots and the swarm of boids. \((t = 0)\).

(b) The car-like mobile robots avoiding the swarm of boids at \(t = 300\) units.

(c) The car-like mobile robots avoiding the swarm of boids at \(t = 400\) units.

(d) The final posture of the car-like mobile robots and the position of the swarm at \(t = 500\) units.

Figure 5.20: There are \(n = 15\) boids (shown in red), each with bin size 3, randomly positioned at the initial time \(t = 0\). The horizontal and vertical axes give the coordinates \(x_i(t)\) and \(y_i(t)\) as \(z_1(t)\) and \(z_2(t)\), respectively, for each individual \(i\) at time \(t \geq 0\). The grey lines show the trajectories and of the individuals. The path of the centroid is given by the green line. The car-like agents are shown in blue color. There are 5 randomly placed obstacles shown in blue.
Table 5.5: Scenario 2. Numerical values of initial state, constraints, and control and convergence parameters. There are 15 boids and 3 car-like mobile robots.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial state of the car-like robot</td>
<td>$(x_{v3}, y_{v3}) = (460, 250)$</td>
</tr>
<tr>
<td></td>
<td>$v_{v1} = v_{v2} = v_{v3} = 3$</td>
</tr>
<tr>
<td></td>
<td>$\omega_{v1} = \omega_{v2} = \omega_{v3} = 0.3$</td>
</tr>
<tr>
<td>Constraints</td>
<td></td>
</tr>
<tr>
<td>Target</td>
<td>$(p_{31}, p_{32}) = (460, 2500)$ and radius $r_{t2} = 10$</td>
</tr>
<tr>
<td>Parameters for the boids</td>
<td></td>
</tr>
<tr>
<td>Coupling/inter-individual obstacle avoidance</td>
<td>$\beta_{ij} =$random$[3, 5]$, for $i, j = 1, \ldots, 15$, $i \neq j$</td>
</tr>
<tr>
<td>Avoidance of agents by boids</td>
<td>$\sigma_{ik} = 1.5$, for $i = 1, \ldots, 15$ and $k = 1, 2, 3$</td>
</tr>
<tr>
<td>Convergence</td>
<td>$\alpha_{v1} = \alpha_{v2} = 0.05$, for $i = 1, \ldots, 15$</td>
</tr>
<tr>
<td>Parameters for the car-like robots</td>
<td></td>
</tr>
<tr>
<td>Inter-individual avoidance</td>
<td>$\varphi_{kl} = 1.2$, for $k, l = 1, 2, 3$, $k \neq l$</td>
</tr>
<tr>
<td>Modulus bound on velocity</td>
<td>$\xi_{ku} = 0.001$, for $i, r = 1, 2, 3$</td>
</tr>
<tr>
<td>Avoidance of fixed obstacles</td>
<td>$\lambda_{kr} = 0.5$, for $i = 1, 2, 3$, and $r = 1, \ldots, 5$</td>
</tr>
<tr>
<td>Convergence</td>
<td>$\delta_{k1} = \delta_{k2} = 5000$, for $k = 1, 2, 3$</td>
</tr>
</tbody>
</table>

5.11 Concluding Remarks

Ultimately, the design and control of a decentralised motion planner for multi-tasking of multi-vehicle systems is a complex, computer intensive yet an interesting problem. In this chapter, we have presented a set of continuous acceleration control laws and a set of continuous velocity control laws that successfully tackle the problem of formation control of car-like mobile robots moving in an environment cluttered with swarm of boids pertaining to a certain emergent behavior and fixed obstacles. The generalised controllers, extracted from the Lyapunov-based control scheme (LbCS), enabled collision-free trajectories from initial to desired final states within a constrained environment, while satisfying the intimately couple nonholonomic constraints of the car-like mobile robots, and the kinodynamic constraints associated with the system. To the author’s knowledge, this is the first time in literature whereby moving car-like mobile robots are modeled together with swarm of emergent behaviors and successfully maintained within the Lyapunov-based control scheme via Lyapunov-like functions.

The seminal strategy of maintaining the swarm as a cluster was based on the cohesiveness of the model and together as one, the swarm of boids are able to tackle the notion of motion planning and control. For instance, obtaining food, avoiding predators, etc.

Moreover, the synthesized control laws guaranteed practical stability of the boids and the car-like mobile robots. The efficiency of the proposed control laws were also verified numerically and visually via computer simulations of two different scenarios. Moreover, the computer simulations demonstrated the successful amalgamation of the minimum distance technique
within the framework of the LbCS. Here, only the swarm that was closest to the car-like robots avoided it successfully without any collision and the other members of the swarm continued exhibiting their unique behavior.
Conclusion

"It is not the strongest of the species that survives, nor the most intelligent that survives. It is the one that is most adaptable to change."

Charles Robert Darwin (1809 - 1882)

In this research, we have utilized one of the most powerful tools for practical stability analysis of systems governed by first-order differential equations. The method that was deployed was provided by Lakshmikantham, Leela and Martynyuk [69] which used the second or direct method of Lyapunov and its owes its applicability to both practical and theoretical problems because of the qualitative information it is able to provide on the behavior of the solutions to the problems. The applicability of this method has been verified by computer simulations which show that this method is very useful for collision avoidance problems especially in the field of robotics. However, failure to have a general framework for the construction of the Lyapunov functions constitutes its major weakness. This is because many fail to find a suitable function to analyze the stability of a given system. Though one may not find a useful or productive function, stability can be analyzed by other methods.

Previous work on swarm modelling suggests that swarming could arise out of the interplay between a long-range attraction and a short-range repulsion between individuals in the swarm, with the centroid being the center of attraction. This study showed that the concept of practical stability based on the Direct Method of Lyapunov was particularly suited to the investigation of swarming since, in general, the centroid could be unstable and non-stationary, yet the behavior of the swarm would be acceptable within its vicinity. Computer simulations not only confirmed this basic feature of collective behavior but also exhibited more complex dynamics such as self-organized oscillatory and random walk-like motions. Further, we showed how the basic model could be extended to include fixed obstacles. The system parameters played the major role in inducing the emergent collective behaviors.

The research presented in this thesis showcases the theoretical development and modelling of a swarm of boids via kinematic equations. The Lyapunov-based control scheme (LbCS) of [114], essentially an artificial potential fields method, was used to address the motion planning and control problem of swarm of boids operating in constrained environments. This control
scheme provides an easy but efficient platform for deriving continuous time-invariant velocity and acceleration controllers of the swarm of boids and the vehicular agents, respectively. The scheme also offers an extended degree of flexibility by taking into account all swarm constraints such as limitations on velocity, restrictions imposed by boundary conditions, an obstacle-ridden workspace, and point and posture stabilities.

Chapter 1 presents an overview of the collective behavior exhibited by swarms and provides a review of the two fundamentally different approaches of swarming behavior, namely, (i) spatial approach; and (ii) non-spatial approach. We then present a comprehensive review on the work done by researchers in the different fields of robotics pertaining to swarming. This includes, (1) the biological literature; (2) the physicists literature; and (3) the engineering literature on swarming. The flocking rules of Reynolds is then explained explicitly followed by a comparative analysis of holonomic and nonholonomic systems. Finally, we discuss the concept of practical stability when applied to Lagrangian swarm models and its applicability to real life situations.

Chapter 2 provides a review and history of the LbCS and outlines an algorithmic representation of the scheme. As an illustration, we consider the motion planning problem of a point-mass confined to a two-dimensional workspace, and construct an attractive potential field function for target attraction and a repulsive potential field function for the avoidance of elliptic obstacles. We also provide a graphical representation of the total potentials. Then, we give a brief overview of the Direct Method of Lyapunov, an integral component of the LbCS, and provide the definition of stability in the Lyapunov sense. We end the chapter by formulating the construction of a three-dimensional swarm model and providing the theoretical background of practical stability provided by Lakshmikantham, Leela and Martynyuk [69].

Chapter 3 utilizes the LbCS to derive a set of continuous time-invariant velocity control laws to tackle the multi-task problem of navigation of a swarm of boids. Via a Lyapunov-like function, we developed an attractive-repulsive swarm model with two components: (1) the first component allowed each individual to seek and converge to the centroid, a behavior that is a form of a long-range attraction between individuals; and (2) the second component controlled the avoidance between individuals. We showed that the model is a gradient system that is practically stable about the centroid via Lakshmikantham, Leela and Martynyuk [69]. This implies that we could get a congregation of individuals about their centroid, forming cohesive and well-spaced swarms. The effectiveness of the LbCS and the resulting nonlinear controllers was verified through computer simulations of the swarm of boids operating in virtual environments.

Chapter 4 generalizes the results of Chapter 3 and presents a set of distributed continuous time-invariant velocity control laws, extracted from the LbCS, to showcase the emergent collective behavior exhibited by the swarm of boids in the presence of fixed and moving obstacles in a constrained environment. The design of a distributed control system for the emergent swarm behaviors is a complex, computer intensive, yet an interesting problem. Again, the effectiveness of the proposed control laws was demonstrated via computer simulations of various scenarios. This chapter marks a starting point in further developing Reynold’s pioneering approach [109], to guarantee practical stability of the emergent swarm behaviors.
Chapter 5 showcases a new set of continuous acceleration control laws, derived from the LbCS, that addresses the problem of formation control of car-like mobile robots moving in an environment cluttered with fixed and moving obstacles. In this chapter, the swarm of boids is presented as moving obstacles for the car-like mobile robot, whereby we provide a one way avoidance scheme. This is the hallmark of the chapter. The individual swarms are able to come to the center and flock whilst avoiding stationary as well as moving obstacles, that is, the car-like mobile robots in the workspace. Again, the synthesized control laws in conjunction with the Direct Method of Lyapunov was used to construct the Lyapunov-like function and the practical stability of the system was guaranteed via Lakshmikantham, Leela and Martynyuk [69]. The efficiency of the proposed control laws was verified via computer simulations of various scenarios.

6.1 Suggestions for Future Work

The work from this thesis may be extended in the following directions:

(1) We will attempt to apply the results in this paper to the control of robot swarms by creating, for instance, a kinematic model of an individual robot in a swarm and constructing its instantaneous velocity along the method expounded in this thesis.

(2) Apply the Lyapunov based control scheme to other biologically inspired applications such as swarm of car-like mobile robots.

(3) Extending the theoretical techniques of practical stability to other robotic systems and providing theoretical proofs for practical stability of the system.

(4) Apply the findings of the thesis to controlling unmanned aerial vehicles, UAV’s. In general, they have more difficult controls required for navigation purposes.

(5) A major future undertaking is the possibility of validating the theoretical results presented in this thesis via experiments, which perhaps, will qualitatively verify the practicality of the results for real-life applications.
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