THE UNIVERSITY OF THE SOUTH PACIFIC
LIBRARY

Author Statement of Accessibility- Part 2- Permission for Internet Access

Name of Candidate : TEUKAVA FINAU.

Degree : MSc in Mathematics

Department/School : Math Department / SCIM.

Institution/University : USP.

Thesis Title : An investigation of Ideals in Construct Banach Algebra Theory.

Date of completion of requirements for award :

1. I authorise the University to make this thesis available on the Internet for access by USP authorised users. [✓ Yes]/No

2. I authorise the University to make this thesis available on the Internet under the International digital theses project [✓ Yes]/No

Signed:

Date: 30th March 2011.

Contact Address

Liahona High School
Tonga.

Permanent Address

Liahona High School
Tonga.

An Investigation of Ideals in Constructive Banach Algebra Theory

A thesis presented to
The University of the South Pacific
in partial fulfilment of the thesis requirement
for the degree of

Master of Science

by

Teukava Finau

The University of the South Pacific

March, 2011

Declarations

Statement by Author

I certify that this thesis is my own work except those sections and results which have been explicitly acknowledged. I also certify that this thesis has not been previously submitted for a degree at any other institution or university.

Signature: .... Date: 30 March 2011.
Name: Mr. Teukava Finau
Student ID: S11016327

Statement by Supervisor

The research in this thesis was performed under my supervision and to my knowledge is the sole work of Mr. Teukava Finau.

Signature: RSH .... Date: 30March2011
Name: Dr. Robin S. Havea
Designation: Senior Lecturer in Mathematics

Abstract

In this thesis, we investigate questions about the structure of ideals in a separable, commutative, unital complex Banach algebra. Everything is to be carried out from a constructive point of view particularly within the framework of Bishop constructive mathematics, that is doing mathematics using intuitionistic logic. Questions regarding the relationship between proper, quasi–maximal, and maximal ideals are to be investigated. Furthermore, Brouwer's Continuity Principle together with his well known Fan Theorem (for detachable bars) are considered in the discussion.

Dedicated to my wife and my son:

Ana Siale Fehoko Finau and Teukava Finau Jr.

Acknowledgements

I would like to take this opportunity to sincerely thank all those who have contributed to the production of this thesis.

My special thanks go to my Supervisor, **Dr. Robin Siale Havea**, Senior Lecturer in the School of Computing, Information, and Mathematical Sciences, University of the South Pacific for introducing me to Constructive Banach Algebra Theory. I would also like to express my appreciation for his copious and careful criticism, motivation and help during my research.

My sincere gratitude goes to the New Zealand Government (NZAID) for the financial support towards my study, without which this work would not have been possible.

I wish to acknowledge the librarians of University of South Pacific for their support and help throughout the entire time I conducted this research.

I am greatly indebted to my family, to my mother, Ana Paongo Finau for her support and encouragement from home. I would also like to express my thank to my parents inlaw, Sitiveni and Kilisitina Fehoko for the laptop and the financial support for this research and study. Lastly my heartful thanks to my wife, Ana Siale Finau, and my son, Teukava Finau Jr., for accompanying me here in Fiji during my study, and for my wife, who undauntedly endured several late hours of my work with subsequent neglect of family duties. All things considered, without her persistent support this thesis would never have been finished.

Contents

Chapter 1

Introduction

The constructive approaches to mathematics arose during the period of what was felt to be functional crisis in the early part of the last century. Each critiqued an essential logical aspect of classical mathematics, namely, concerning the unrestricted use of the *Law of Excluded Middle* (**LEM**), $P \vee \neg P$ for any meaningful statement P, on the other hand and of apparently circular "impredicative" definitions on the other. But the positive redevelopment of mathematics along a constructive manner, did not emerge as really viable alternatives to classical, set-theoretically based mathematics until the 1960s.

There is a relatively massive amount of information available to which this investigation was based. To be specific, the focus of this investigation is on theoretical interrelationships between major formal systems for constructive and classical mathematics particularly in the context of ideals in a commutative (complex) ***Banach algebra***: an algebra $\mathcal{B}$ over the field of complex numbers, with a multiplicative identity e, which is endowed with a norm $\|\cdot\|$ satisfying the following conditions.

(i) $\|e\| = 1$,

(ii) $\mathcal{B}$ is a Banach space relative to the norm $\|\cdot\|$,

(iii) $\|xy\| \leq \|x\|\|y\|$ for all x and y in $\mathcal{B}$.

1.1 A brief history of constructive mathematics

Although luminaries such as Leopold Kronecker had advocated a constructive approaches to mathematics in the nineteenth century, the story of modern constructivism really begins with the publication in 1907, of the doctoral thesis [18], in which the Dutch mathematician L.E.J Brouwer introduced his Intuitionistic mathematics **(INT)** as an alternative to traditional classical mathematics **(CLASS)**. He devoted the best part of his mathematical life to a penetrating criticism of the constructive inadequacy of the mathematics of his day. According to Brouwer, 'mathematical objects are free–creations of human mind' ([5], page 7) independent of both logic and language, and a mathematical object comes into existence precisely when it is constructed. Such a belief naturally leads to a rejection of existence proofs by contradiction and a consequent sceptism about meaning of many of the theorems of **CLASS**. Not surprisingly, Brouwer's views met with at best indifference, and at worst hostility, from the last majority of his peers for whom the elimination of nonconstructive arguments, with all their apparent power and fruitfulness, was too great a price to pay for a clarification of the meaning of mathematics.

If we adhere to the principle that "existence" should always be interpreted constructively, then we are forced to dispense with the unrestricted use of **LEM**. Recognising this consequence of his philosophical views, Brouwer went far and claimed:

> *The belief in the universal validity of the principle of the excluded third in mathematics, is considered by the intuitionists as a phenomenon of the history of civilization of the same kind as the former belief in the rationality of π, or in the rotation of the firmament about the earth.* ([29], page 7)

Subsequently, he introduced into **INT** some principles that led to results apparently contradicting aspects of **CLASS**. For example, in Brouwerian Intuitionism, the real numbers are treated in some way or another as Cauchy sequences of rational understood as ‘free choice sequences’[1]. Brouwer’s idea concerning these seems to be that one has only a finite amount of information about such sequences at any given time. That was a kind of argument for the continuity conclusion, namely, that any constructive function of choice sequences must be continuous, even more:

> **Brouwer’s Theorem.** *Every function on the closed interval* $[a, b]$ *is uniformly continuous.*

This, in the face of it, is indirect contradiction to classical mathematics, but once it is understood that Brouwer’s theorem must be explained differently via the intuitionistic interpretation of the notions involved, an actual contradiction is avoided.

Nevertheless, there was, and remains, a commonly held belief that too much mathematics has to be given up in order to accommodate Brouwer’s ideas particularly the rejection of **LEM**. The famous Hilbert expressed his disagreement with Brouwer in words both forceful and memorable.

> *Forbidding a mathematician to make use of the principle of excluded middle is like forbidding an astronomer his telescope or a boxer the use of his fists.* (see [9])

Despite continuing opposition, Intuitionism survived and a new constructive approaches to mathematics arose. In 1948–1949 in the former Soviet Union, A. A. Markov initiated a programme of recursive constructive mathematics (**RUSS**). This is primarily mathematics using intuitionistic logic based on the Church–Markov–Turing thesis—all (computable) partial functions from $\mathbb{N}$ to itself are recursive—and this approach had led to a number of technical successes. Furthermore, **RUSS** does not use any of Brouwer’s nonlogical intuitionistic principle. In fact, it could not since it produces result that are false if interpreted directly within **INT**. For instance, in **RUSS**, there exists a continuous real-valued map on $[0, 1]$ onto $(0, 1]$

---

[1] See page 62 of [19] for detail discussion of (free) choice sequences in connection to lawlike and lawless sequences.

that has infimum equal to 0. Once again, one should not overreact to the apparent conflict with classical mathematics: the last of these results should really be interpreted as saying that there exists a recursively uniformly continuous recursive function f from the closed interval $[0, 1]$ of the recursive real line onto the recursive interval $(0, 1]$ that has infimum equal to 0. Put this way, the result does not conflict with **CLASS**; indeed it is a result of **CLASS**, since the proof within **CLASS** that does not use such nonconstructive logical principle as **LEM**. Furthermore, **RUSS** had been identified as being very successful with their approach in complexity theory and its applications to computer science than in the practice of constructive mathematics.

By the mid–1960s, constructive mathematics was, when compared with its classical counterpart, virtually stagnant. This situation changed in 1967 when Errett Bishop published his monograph *Foundation of Constructive Analysis* [2]. This book and its offsprings [3] and [5] represent the most far reaching and systematic presentation of constructive analysis to date. However, this variety of constructive mathematics is often dubbed as the 'Bishop–style constructive mathematics' (**BISH**) which is distinctive in a number of ways when compared to other schools of constructivism.

Bishop had been working in classical analysis and had made important contributions to that subject over a long period of time. But then, he had some radical change of views about classical analysis and felt that it had to be redeveloped on the entirely constructive grounds. By reconstructing a significant part of modern analysis, Bishop was able to revitalize the subject. In [2], without resorting to either Brouwer's principles or the formalism of recursive function theory, Bishop revealed by thorough–going constructive means that a vast amount of constructive mathematics covering elementary analysis, metric and normed spaces, abstract measure and integration, the spectral theory of selfadjoint operators on Hilbert space, Haar measure, duality on locally compact groups, and Banach algebras can be constructivised.

Beside these works in constructive analysis, a substantial amount of classical algebra and analysis have been redeveloped in the Bishop style approach. Interestingly, Bishop criticised non–constructive classical mathematics by referring to it as a “scandal”, particularly because of its “deficiency in numerical meaning”. What he simply meant was that if you say something exists, you ought to be able to produce it, or in other words, provide an algorithm or “finite routines” of how to find it. And if you say there is a function which does something on the real numbers then you ought to be able to produce a machine which calculates it out at each number.

Bishop kept strictly to the interpretation of “existence” as “computability”. His refusal to pin down the notion of algorithm led to criticism, particularly from philosophers of mathematics and from those committed to the Church–Markov–Turing thesis. But this very imprecision enable Bishop’s work to have a variety of interpretations: his results are valid in **CLASS**, **INT**, and **RUSS**. Indeed, from purely formal view point, each of **INT**, **RUSS**, and **CLASS** can be regarded as **BISH** plus some additional principles:

- **INT** = **BISH** + Brouwer’s continuity principle and fan theorem;
- **RUSS** = **BISH** + Church–Markov–Turing thesis;
- **CLASS** = **BISH** + **LEM**.

One consequence of this multiplicity of interpretations is that we can often demonstrate that certain proportion P are independent of **BISH**; that is, neither P nor $\neg P$ can be proved within **BISH**. For example, since “*every mapping from [0,1] into* $\mathbb{R}$ *is uniformly continuous*” is a theorem of **INT**, and “*there exists from [0,1] into* $\mathbb{R}$ *that is not continuous*” is a theorem of **RUSS**, and since both **INT** and **RUSS** are formally consistent with **BISH**, within **BISH** we cannot expect either to prove that continuous map of $[0,1]$ into $\mathbb{R}$ is uniformly continuous or to construct an example of a real–valued function that is defined, but not uniformly continuous, on $[0,1]$.

Over the years since the publication of Bishop's book, [2], it became clear to a number of researchers that in essence **BISH** is simply *mathematics with intuitionistic logic* together with some appropriate set–theoretic foundation. As we know, working with intuitionistic logic automatically bars noncomputational steps. As long as we keep strictly to intuitionistic logic, having make sure that our set–theoretic principles do not inadvertently imply **LEM** or some other nonconstructive proposition, the mathematic we develop turns out to be predictive, in the sense that every proof implicitly shows that if we perform certain calculations, we shall achieve certain results. Throughout in the discussions to follow, we speak of *constructive mathematics* or **BISH** to consistently mean (in the Richman sense) "*doing mathematics using intuitionistic logic*". In fact, the approach in this thesis will be within the framework of **BISH**.

1.2 Logic and constructive interpretations

The following is the BHK–interpretation[2]. Let P and Q be any given statements.

- ▶ $\wedge$ (**and**) To prove $P \wedge Q$ (P and Q), we must have a proof of P and a proof of Q.

- ▶ $\vee$ (**or**) To prove $P \vee Q$ (P or Q), we must have either a proof of P or a proof of Q.

- ▶ $\Rightarrow$ (**implies**) To prove $P \Rightarrow Q$ (P implies Q) means there is an algorithm that transforms a proof of P into a proof of Q.

---

[2] These interpretations are attributed to Brouwer, Heyting, and Kolmogorov.

- $\neg$ (**not**) We interpret $\neg P$ (not P) as $P \Rightarrow Q$, where Q is a contradiction such as 0=1.

- $\exists$ (**exists**) To prove $\exists a P(a)$ (There exists a such that a has the property P), we must compute a and demonstrate that $P(a)$ holds.

- $\forall$ (**for all**) A proof of $\forall a \in AP(a)$ (For all a in A, a has the property of P) is an algorithm that, applied to each element a of A and to the data showing that a belongs to A, prove that $P(a)$ holds.

As I mentioned earlier, these constructive interpretations enabled Heyting in [24] to produce a complete list of the axioms of intuitionistic logic which is regarded as a generalization of classical logic [26]. In the next section, we shall see the impact of intuitionistic logic, which leads to the rejection of some trivial principles of classical mathematics which is based on classical logic.

The interpretation of $P \wedge Q$ is similar to the classical treatment. Classically, to prove $P \vee Q$ it suffices to establish $\neg(\neg P \wedge \neg Q)$ but proving the latter is not enough to prove the former in constructive mathematics. Why? Generally, in constructive mathematics it is not possible to decide, from a proof of $\neg(\neg P \wedge \neg Q)$, which of the alternatives P, Q holds. The constructive interpretation of disjunction $\vee$ is well tied to the very notion of decidability and undoubtedly one of the features of constructive mathematics is being able to make decision and the constructive interpretation of $\vee$ captures it all.

To prove $\exists x \in AP(x)$ in **CLASS**, it suffices to show that $\neg \forall x \neg P(x)$. Classical existence is equivalent to the impossibility of nonexistence. On the other hand, to prove $\exists x \in AP(x)$ constructively, we must construct an object ξ (at least in principle), show that ξ satisfies the conditions for membership of A, and then show that $P(\xi)$ holds.

1.3 Roots of nonconstructivity

In this section, we discuss some of the common principles that bear the seeds of nonconstructivity in mathematics. Most of these principles are classically trivial. We will give some well known examples shown in [22], demonstrating how untrustworthy they are in Constructive mathematics. Upon rejecting these principles, it follows that any statement proved to be equivalent to, or implying, any of them is regarded as essentially nonconstructive.

The first principle, which we have met earlier, is.

- **Law of Excluded Middle (LEM)**: For any given statement P, either P is true or P is false.

Bearing in mind the constructive interpretation for logic discussed above, we see that the rejection of **LEM** is closely connected with the fact that the property:

$$\forall n P(n) \vee \neg \forall n P(n)$$

need not hold in constructive mathematics even when $P(n)$ is a decidable property of the natural number n. Much of the reasoning we normally encounter in **CLASS** is based on **LEM**. Applications of **LEM** make life easier, especially in the case of existential proofs. Consider the following classical theorem.

There exist irrational number r and s such that r^s is rational.

Following a simple argument given by Bishop ([3], page 6), consider the real number $\sqrt{2}^{\sqrt{2}}$. By virtue of **LEM**, either $\sqrt{2}^{\sqrt{2}}$ is rational or $\sqrt{2}^{\sqrt{2}}$ is irrational. In the former case, if $\sqrt{2}^{\sqrt{2}}$ is rational, then simply take $r = s = \sqrt{2}$ and we are done. In the latter case, if $\sqrt{2}^{\sqrt{2}}$ is irrational, then take $r = \sqrt{2}^{\sqrt{2}}$ and $s = \sqrt{2}$. Then r and s are irrational and $r^s = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = 2$ which is rational. A closer look at this proof shows that under **LEM** we have proved there are numbers r and s satisfying the claim but there is no hint at all on how to actually find the two numbers.

Recall that a ***binary sequence*** is a finite routine that assigns to each positive integer an element of $\{0, 1\}$. The next three principles deal with such sequences and are classically trivial and special cases of **LEM**.

- **Limited Principle of Omniscience (LPO)**: If (a_n) is a binary sequence, then either $a_n = 0$ for all n or else there exists n such that $a_n = 1$.
- **Weak LPO (WLPO)**: For any binary sequence (a_n), either $a_n = 0$ for each n, or else it is impossible that $a_n = 0$ for each n.
- **Lesser LPO (LLPO)**: If (a_n) is a binary sequence containing at most one term equal to 1, then either $a_{2n} = 0$ for each n, or else $a_{2n+1} = 0$ for each n.

It appears that Brouwer was responsible for first drawing attention to the nonconstructive nature of **LPO** and **LLPO** [29, 30] though under different names but we have chosen to use the names given by Bishop in [4]. None of these three omniscience principles can be derived within Heyting arithmetic, and each is false, even classically, in recursive mathematics. (See Chapters 1, 3, and 7 of [16])

The following generalisation is due to Richman ([27], page 135) which is a hierarchy of ever weaker omniscience principles. Notice that **LLPO** is the special case when $N = 2$.

- **LLPO**$_N$ $(N = 2, 3, 4, \dots)$ If (a_n) is a binary sequence with at most one term equal to 1, then there exists j, where $0 \leq j \leq N - 1$ such that $a_{kN+j} = 0$ for all k.

It is well known ([9], page 145) that each of the following statements is equivalent to **LPO** and therefore considered highly nonconstructive.

- $\forall x \in \mathbb{R}(x = 0 \vee x \neq 0)$.
- **Law of Trichotomy**: $\forall x \in \mathbb{R}(x < 0 \vee x = 0 \vee x > 0)$.

- **Least–upper–bound Principle**: Each nonempty subset of $\mathbb{R}$ that is bounded has a least upper bound.
- Every real number is either rational or irrational.

Similarly, each of the following is equivalent to **LLPO** ([9], page 146).

- $\forall x \in \mathbb{R}(x \geq 0 \vee x \leq 0)$.
- If $x, y \in \mathbb{R}$ and $xy = 0$, then $x = 0$ or $y = 0$.
- **Intermediate Value Theorem**: If $f : [0, 1] \to \mathbb{R}$ is a continuous function with $f(0) < 0 < f(1)$, then there exists $x \in (0, 1)$ such that $f(x) = 0$.

A more controversial one is

- **Markov's Principle (MP)**: If (a_n) is a binary sequence for which it is contradictory that all terms be 0, then there exists n such that $a_n = 1$.

Though accepted and freely used in **RUSS**, **MP** is considered a form of unbounded search and is considered highly nonconstructive in both **INT** and **BISH**. Furthermore, **MP** is regarded as a form of unbounded search that cannot be proved in Heyting arithmetic[3] (see pages 137–138, [16]).

Constructive Mathematics is often identified, wrongly, with its rejection of the full–blooded

- **Axiom of Choice (AC)**: If A, B are sets, and S is a nonempty subset of $A \times B$ such that for each $a \in A$ there exists $b \in B$ with $(a, b) \in S$, then there exists a (*choice*) function $f : A \to B$ such that $(a, f(a)) \in S$ for all $a \in A$.

From its introduction by Zermelo in 1908, **AC** was regarded with unease by many mathematicians and it was rejected outright by the intuitionists. However, it was later in 1978 that Goodman and Myhill in [20] showed that **AC** entails **LEM**. The

---

[3] Peano arithematic with intuitionistic logic.

proof is a very clever one as presented in [20]. Let P be any constructively meaningful statement and define the set $A = \{s, t\}$ together with the equality relation given by

$$s = t \quad \Leftrightarrow \quad P \text{ holds.}$$

Now, consider the set $B = \{0, 1\}$ with the standard equality, and let

$$S = \{(s, 0), (t, 1)\} \subset A \times B,$$

with the equation relation derived from those on A and B, that is

$$(x, y) =_{A\times B} (x_1, y_1) \qquad \Leftrightarrow \qquad x =_A x_1 \text{ and } y =_B y_1.$$

Assume that there exists a function $f : A \to B$ such that $(x, f(x)) \in S$ for all $x \in A$. To complete the proof, we simply make the following observations.

▷ If $f(s) = 1$ or $f(t) = 0$, then $s = t$ and hence P holds.

▷ If $f(s) = 0$ and $f(t) = 1$, then $\neg(s = t)$ and therefore $\neg P$ holds.

Though **AC** cannot be used in any coherent constructive mathematics, most mathematicians in constructive mathematics use the **principle of countable choice**—which is $A = \mathbb{N}$ in **AC**—and the **principle of dependent choice**: If $a_0 \in A$, and if for each $a \in A$ there exists $a' \in A$ such that $P(a, a')$, then there exists a mapping $f : \mathbb{N} \to A$ such that $f(0) = a_0$ and $P(f(n), f(n + 1))$ for each $n \in \mathbb{N}$.

We end this section with the remark that working in Constructive mathematics means working with fewer axioms and principles which is contrary to that of **CLASS**. The rejection of many classically innocent, trivial principles brings extra challenge to a practising constructive mathematician. This does not mean that constructive mathematics is an attempt to replace everything in **CLASS** but one can view constructive mathematics as being more focussed as a mathematical *revival* where numerical meaning and computational content are central and essential to the whole mathematical affair. In the preface to his book [2], Bishop wrote:

> *We are not contending that idealistic mathematics is worthless from the constructive point of view. This would be as silly as contending that unrigorous mathematics is worthless from classical point of view. Every theorem proved with idealistic methods presents a challenge: to find a constructive version, and to give it a constructive proof.*

1.4 Role of Brouwerian examples

In this section we briefly look at how does one identify through formal justification that a given statement P is highly nonconstructive. To be specific, one shows that a given proposition P is nonconstructive by means of 'examples' in which it is shown that P entails, or is equivalent to, some highly known nonconstructive principles. Such demonstration is very useful particularly when one has to work out a constructive version of a nonconstructive result.

For illustration purposes, suppose that we can prove **LPO** constructively. Then, applying it to the binary sequence $\mathbf{a} = (a_n)_{n=1}^{\infty}$ defined by

$$a_n = \begin{cases} 0 & \text{if } 2k \text{ is a sum of two primes for } \quad 2 \leq k \leq n+1, \\ 1 & \text{otherwise} \end{cases}$$

we can either prove the Goldbach Conjecture P—every even integer greater than 2 is a sum of two primes—or else providing an explicit counterexample by giving an integer greater than 2 that is not a sum of two primes; unless we have such construction, as mentioned earlier we are not entitled to assert $P(\mathbf{a}) \vee \neg P(\mathbf{a})$ within **BISH**.

In the preceding illustration, we call $\mathbf{a}$ a ***Brouwerian example*** of a binary sequence for which $P(\mathbf{a}) \vee \neg P(\mathbf{a})$ does not hold, or that $\mathbf{a}$ is a ***Brouwerian counterexample*** to the statement $P(\mathbf{a}) \vee \neg P(\mathbf{a})$. It should be pointed out that a Brouwerian counterexample is not a counterexample in the usual sense but rather a demonstration as an evidence that there is no hope that a statement would admit a constructive proof. More generally, a Brouwerian counterexample to an assertion P is a demonstration by means of a proof that P entails some principle that is unacceptable, or at least highly doubtful and questionable, in the context of constructive mathematics.

We now look at some few Brouwerian examples. The first two examples are adapted from the presentation given by Bridges and Dediu in [9]. Our first example is from the real number line $\mathbb{R}$.

Brouwerian Example 1.4.1 We show that the statement

> *If every real number is either rational or irrational.*

entails **LPO**.

Let $(a_n)_{n=0}^{\infty}$ be an increasing binary sequence, and define a real number by

$$x = \sum_{n=0}^{\infty} \frac{1 - a_n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{a_n}{n}.$$

Suppose that either x is rational or x is irrational. To establish **LPO**, we proceed as follow.

- ▷ If x is rational, then $|x - \mathrm{e}| > 0$. So there exists N such that

$$\sum_{n=0}^{N} \left(\frac{1}{n!} - \frac{1 - a_n}{n!} \right) > 0.$$

  Hence $a_n = 1$ for some $n \leq N$.

- ▷ If x is irrational, then, clearly, $a_n = 0$ for all n. ☺

Our next example is from the complex plane $\mathbb{C}$.

Brouwerian Example 1.4.2 We show that the statement

> *For each complex number z there exists $\theta \in [0, 2\pi)$ such that $z = |z|\mathrm{e}^{\mathrm{i}\theta}$, and such that if $\theta \neq 0$, then $z \neq 0$.*

entails **LPO**.

Let $(a_n)_{n=1}^{\infty}$ be an increasing binary sequence with at most one term equal to 1. Define a sequence of complex numbers $(z_n)_{n=1}^{\infty}$ such that

$$\begin{aligned} a_n = 0 \quad &\Rightarrow \quad z_n = 0 \\ a_n = 1 - a_{n-1} \quad &\Rightarrow \quad z_k = \frac{1}{n}\mathrm{e}^{\mathrm{i}\pi/2} \qquad \text{for all } k \geq n. \end{aligned}$$

It can be shown that $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence and therefore converges to a limit $z \in \mathbb{C}$. Assume that $z = |z|\mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in [0, 2\pi)$, and such that if $\theta \neq 0$, then $z \neq 0$. Then either

$$\theta < \frac{\pi}{2} \qquad \text{or} \qquad \theta > 0.$$

As in our previous example, we now establish **LPO** through the following observations.

- ▷ If $\theta < \pi/2$, then $a_n = 0$ for all n: for if we suppose that there exists n such that $a_n = 1 - a_{n-1}$, then $z = 1/n\mathrm{e}^{\mathrm{i}\pi/2}$ and therefore $\theta = \pi/2$, a contradiction.
- ▷ If $\theta > 0$ we have $z \neq 0$, so there exists N such that $z_N \neq 0$. Thus $a_n = 1$ for some $n \leq N$. ☺

Our last example is from the context of a Banach algebra and it is an adaptation of that presented in both [5] and [7] which is originally due to Bishop [2].

Let $\mathcal{B}$ be a commutative Banach algebra, and $\mathcal{B}^*$ its dual. A ***character*** of $\mathcal{B}$ is a bounded homomorphism of $\mathcal{B}$ onto $\mathbb{C}$, and that the ***character space*** (or ***spectrum***) of $\mathcal{B}$ is the set

$$\Sigma_{\mathcal{B}} = \{u \in \mathcal{B}^* : u(xy) = u(x)u(y), \text{ for all } x, y \in \mathcal{B}\}.$$

When the context is clear we write Σ to consistently mean $\Sigma_{\mathcal{B}}$.

In constructive mathematics, we define ***compactness*** to mean that the set is *complete* and *totally bounded*[4]. This definition equivalent to the commonly used classical definition that involves extracting finite subcovers though the constructive definition proves to be more useful.

Brouwerian Example 1.4.3 We show that the statement

> *The spectrum of every separable commutative unital Banach algebra is compact.*

implies **WLPO**.

Let $(a_n)_{n=0}^{\infty}$ be an increasing binary sequence. Let $\mathcal{B}$ be the algebra consisting of all sequences $\mathbf{x} = (x_n)_{n=0}^{\infty}$ of complex number for which

$$\|\mathbf{x}\| = \sum_{n=0}^{\infty} (1 - a_n) |x_n| \tag{1.1}$$

exists. We define the elements $\mathbf{x} = (x_n)_{n=0}^{\infty}$ and $\mathbf{y} = (y_n)_{n=0}^{\infty}$ of $\mathcal{B}$ to be equal if $\|\mathbf{x} - \mathbf{y}\| = 0$. Then $\mathcal{B}$ is a Banach space equipped with the norm (1.1). Moreover, if we define the product of any two elements $\mathbf{x}$ and $\mathbf{y}$ of $\mathcal{B}$ by

$$\mathbf{xy} = \left(\sum_{i=0}^{n} x_i y_{n-i} \right)_{n=0}^{\infty},$$

then $\mathcal{B}$ is a Banach algebra with identity $e = (1, 0, 0, \ldots)$. Let

$$\mathbf{z} = \left(1, 2^{-1}, 2^{-2}, 2^{-3}, \ldots\right) \in \mathcal{B}.$$

If $a_n = 1$ for some n, then the character space $\Sigma_{\mathcal{B}}$ consists of the single element $\mathbf{x} \mapsto x_0$. On the other hand, if $a_n = 0$ for all n, then to each complex number ξ with $|\xi| \leq 1$ there corresponds an element u_ξ of $\Sigma_{\mathcal{B}}$ defined by

$$u_\xi(\mathbf{x}) = \sum_{n=0}^{\infty} x_n \xi^n.$$

---

[4] Let (X, ρ) be a metric space. An ϵ ***approximation*** to X is a subset Y of X such that for each x in X there exists y in Y with $\rho(x, y) < \epsilon$. We say that X is ***totally bounded*** if for each $\epsilon > 0$ there exists a finite ϵ approximation.

Suppose that $\Sigma_{\mathcal{B}}$ is compact. Since the mapping $u \mapsto |u(\mathbf{z})|$ is uniformly continuous relative to the weak*–topology on the unit ball of $\mathcal{B}^*$, it maps $\Sigma_{\mathcal{B}}$ to a totally bounded subset of $\mathbb{R}$. Hence

$$R = \sup_{u \in \Sigma_{\mathcal{B}}} |u(\mathbf{z})|$$

exists. Either $R > 1$ or $R < 2$. To establish **WLPO**, we proceed as follow.

- ▷ If $R > 1$, then $a_n = 0$ for all n.
- ▷ If $R < 2$, we cannot have $a_n = 0$ for all n. ☺

1.5 Brief overview of ideals and Banach algebra

In this section, we recall and introduce some key concepts that are directly relevant to discussions to follow in later chapters of the thesis particularly that we'll be working with ideals on commutative Banach algebra. One of the key and interesting results which we'll pay special attention to is the classical proposition that *every ideal is contained in a maximal ideal*. Furthermore, we look at related substitutes for other classical results that directly involved compactness of the spectrum with regard to the standard expression for the spectral norm.

In this thesis, we speak of an 'algebra' to consistently mean a separable commutative (unital) complex Banach algebra with an identity e.

The original constructive development of Banach algebra theory in [2] was based on the notion of a partial ideal which was a substitute for the classical notion of a finitely–generated ideal. The cornerstone of that development was a constructive substitute for the classical result that if I is a proper ideal in R and x is any element in R, then there exists a proper ideal J containing I, and a complex number λ, such that $x - \lambda e \in J$. It is noted that the development by Bishop and Bridges, particularly Theorem (2.1) (on page 453) of [5] is much more natural and elegant than that in the original presentation by Bishop in [2]. It is along these recent approaches to constructive Banach algebra theory that the discussions to follows will align to.

The basic notions of ideal theory are examined constructively in the context of a commutative ring with an identity and an inequality relation by Mines et al. in [25] and in recent works of Bridges particularly [8]. Constructive analogues of classical theorems relating maximality and primeness are proved and well presented by Bridges in [8]; from time to time and whenever necessary we shall refer to this paper of Bridges in Chapters 2 and 3 of this thesis.

When working constructively in Banach algebra, one does not have to be a strict constructivist to appreciate the applicability and relations to aspects of computational mathematics. Because our reasoning is based on intuitionistic logic, one advantage is that the proofs and results have more interpretations than their classical counterparts; that is working in **BISH**, it

- ▷ can be interpreted recursively or intuitionistically,
- ▷ can be translated into any formal system for computable analysis, and
- ▷ is consistent with **CLASS**.

However, the intention of this research (thesis) is to investigate the constructive status of some well known results in Banach algebra, especially ideal, and see what can be done in order to push them into a constructive setting.

In particular, we look at works of Bishop in [2, 3], Bishop and Bridges in [5], Bridges in [6, 8], Bridges and Havea in [10, 11, 13, 14], Bridges et al. in [15], and Havea in [21, 23]. To be specific, this investigation will primarily based on selected works of these authors which can be considered as recent development in the field. We identify key problems concerning ideals in a constructive Banach algebra setting and investigate possible answers to various questions. Some of the key issues to consider include the following.

1. Investigate the best constructive substitute for the classical result which mentioned above that *every proper ideal is contained in a maximal ideal*.

2. Investigate constructively the concepts of: maximal ideals, stability of an ideal, weak maximal ideals, and detachable ideals; how can we use these to recover some of the interesting classical results.

3. Find an answer to the open question: When is a maximal ideal weakly maximal? Constructively the answer is affirmative if the ideal is stable but without stability, it is not that clear.

1.6 Thesis layout

As previously mentioned, the analysis and presentation throughout this thesis are carried out within the framework of Bishop–style constructive mathematics. For details discussion of the techniques of constructive analysis, the reader is kindly referred to [17].

The thesis has four main chapters including this introductory one. The second chapter deals with notions and concepts of maximal ideals, whereas the third chapter continues the investigation of ideals but in the context of a Banach algebra. The last chapter is a short presentation summarising key concluding remarks of this investigation and pointing to possible future work in this particular area.

Chapter 2

Constructive Notions of Maximality for Ideals

In this chapter, we continue the constructive study of rings and ideals by examining two constructively distinct (but classically equivalent) notions of maximality for ideals. Firstly, we recall some constructive concepts which will be necessary throughout the discussions to follow. The presentation throughout in this chapter is in line with the works of Bridges in [8] and Bridges and Havea in [14].

2.1 Preliminary

Let R be a commutative ring with an identity element e equipped with a ***ring inequality*** which is a binary relation $\neq$ that satisfies not only the usual constructive conditions for an inequality, $x \neq y \Rightarrow y \neq x$ and $x \neq y \Rightarrow \neg(x = y)$, but also $e \neq 0$ and the following:

$$\begin{aligned} x \neq y \quad &\Rightarrow \quad x - y \neq 0, \\ x + y \neq 0 \quad &\Rightarrow \quad x \neq 0 \vee y \neq 0, \\ xy \neq 0 \quad &\Rightarrow \quad x \neq 0 \wedge y \neq 0. \end{aligned}$$

Note that as a consequence of the last three conditions we have

$$x \neq y \Leftrightarrow \forall z \in R(x + z \neq y + z).$$

The ***complement*** of a subset S of R is the set

$$\sim S = \{x \in R : \forall y \in S(x \neq y)\}.$$

In general, this is not the same constructively as the ***logical complement***

$$\neg S = \{x \in R : x \notin S\}$$

of S.[1] Furthermore, in the context of a metric space (X, ρ), the ***metric complement*** of a subset S of X is

$$-S = \{x \in X : \exists r > 0 \forall y \in S(\rho(x, y) \geq r)\}.$$

If S is a subset of set X with an inequality $\neq$, we say that S is

- $\triangleright$ ***reflective*** if for each $x \in X$ there exists a point b of S, called an S–test point for x, such that if $x \neq b$, then $x \in \sim S$;
- $\triangleright$ ***semidetachable*** if $\forall x \in X(\neg(x \in \sim S) \Rightarrow x \in S)$;
- $\triangleright$ ***detachable*** if $\forall x \in X(x \in S \vee x \in \sim S)$.

If S is reflective and the inequality is tight, then S is semidetachable. If S is nonempty and detachable, then it is reflective: to see this, we obtain an S–test point b for x by setting b equal to x if $x \in S$, and for any point of S if $x \in \sim S$. If X is 'discrete' in the sense that

$$\forall x, y \in X(x = y \vee x \neq y),$$

---

[1] Take for example, in the ring $\mathbb{R}$ of real numbers, or more generally in any Banach algebra, the inequality is defined by

$$x \neq y \Leftrightarrow \|x - y\| > 0$$

and, in the absence of **MP**, is stronger than the denial inequality $\neg(x = y)$.

then being reflective implies detachability, since for a reflective set S we can check the alternatives $x \in S \vee x \in\sim S$ by comparing x with one of its S–test points.

We call the ring R

- ▷ ***discrete*** if $\forall x \in R(x = 0 \vee x \neq 0)$;
- ▷ ***quasidiscrete*** if $\forall x \in R(x \text{ invertible} \vee x \neq e)$.

If R is discrete, then for all $x, y \in R$ either $x = y$ or $x \neq y$. Clearly, a discrete ring is quasidiscrete. A Banach algebra $\mathcal{B}$ is quasidiscrete, since for each $x \in \mathcal{B}$ either $x \neq e$ or $\|e - x\| < 1$, and in the latter case x is invertible; but if even the Banach algebra $\mathcal{B}$ were discrete, then we would be able to prove **LPO**. This is thoroughly discussed by Bridges and Richman in first chapter of [16].

We denote by $\langle S \rangle$ the ideal generated by a subset S of R. In the special case where $S = T \cup \{x\}$ for some $T \subset R$ and $x \in R$, we write $\langle T, x \rangle$ for $\langle S \rangle$; if, further, $T = \varnothing$, we write $\langle x \rangle$ rather than $\langle \{x\} \rangle$.

We say that an ideal I of R is

- ▷ ***proper*** if $e \in\sim I$;
- ▷ ***maximal*** if it is proper and for each $x \in\sim I$ the ideal $\langle I, x \rangle$ equals R;
- ▷ ***weakly maximal*** if it is proper, and for each $x \in R$, if $\langle I, x \rangle$ is a proper ideal, then $x \in I$;
- ▷ ***stable*** if $\sim\sim I = I$.

Note that a proper ideal I is weakly maximal if and only if $M = I$ whenever M is a proper ideal that includes I.

Although the notions of "maximal ideal" and "weakly maximal ideal" are classically equivalent, they can be distinguished constructively, as the following Brouwerian example shows. However, we shall see later in Corollary 2.3.7 that every weakly maximal ideal of $\mathbb{Z}$ is maximal.

Brouwerian Example 2.1.1 We show that the statement

> *Every maximal ideal of $\mathbb{Z}$ is weakly maximal.*

entails **LEM**.

In the (discrete) ring $\mathbb{Z}$, let I be ideal generated by the set

$$S = \{n \in \mathbb{Z} : n = 4 \vee (n = 2 \wedge (P \vee \neg P))\},$$

where P is any syntactically correct mathematical statement. Since $\neg(P \vee \neg P)$ is false, $\sim I$ is the set of odd integers and therefore I is maximal. Now, I contains the even integers and that $\langle I, 2\rangle = \langle 2\rangle$ is a proper ideal. If $\langle I, 2\rangle = I$, then $P \vee \neg P$ holds. ☺

2.2 Stable ideals

We begin our exploration of the links between maximality, weakly maximality, and stability with a couple of simple lemmas. These are adaptations of the works appeared in [14].

Lemma 2.2.1 *If I is a proper ideal in R and if $x \in R$ is invertible, then $x \in\sim I$.*

Proof For each $y \in I$ we have $x^{-1}y \in I$, and since I is a proper ideal, that is $e \in\sim I$, so $x^{-1}y \neq e = x^{-1}x$ since x is invertible, therefore $x^{-1}(x - y) \neq 0$; whence $y \neq x$. Since $y \in I$ and $x \in R$ are arbitrary. It follows that $x \in\sim I$. ■

Lemma 2.2.2 *Let I be an ideal in R. Let $p \in I$, $q \in R$, and $x \in R$ be such that $p + qx \in\sim I$. Then $x \in\sim I$.*

Proof For each $y \in I$, and by definition of ideal, qy is in I and therefore $p + qy \in I$, where $p \in I$. Now, since $p + qy \in I$, we have $p + qx \neq p + qy$. It follows that $qx - qy = q(x - y) \neq 0$ and hence $y \neq x$. Therefore $x \in\sim I$. ■

Lemma 2.2.3 *If R is quasidiscrete and I is a proper ideal of R, then for each $x \in \sim\sim I$, the ideal $\langle I, x \rangle$ is proper.*

Proof Given $x \in \sim\sim I$, consider any element $p + qx$ of $\langle I, x \rangle$, where $p \in I$ and $q \in R$. Suppose that $p+qx$ is invertible. Then, by Lemmas 2.2.1 and 2.2.2, $x \in \sim I$, which is absurd. Hence $p + qx$ is not invertible and therefore, by quasidiscreteness, $p + qx \neq e$. Since $p \in I$ and $q \in R$ are arbitrary, we conclude that $e \in \sim \langle I, x \rangle$. Therefore $\langle I, x \rangle$ is proper. ▮

We note that the trick used in the preceding proof to show that $p + qx \neq e$ is used again, twice, in the proof of Proposition 2.2.5 below. But first we deal with the stability of weakly maximal ideals.

Proposition 2.2.4 *If R is quasidiscrete, then every weakly maximal ideal in R is stable.*

Proof First note that $\sim\sim I \supset I$. It remains to establish the other inclusion. Let I be a weakly maximal ideal in R, and $x \in \sim\sim I$. Since I is proper, Lemma 2.2.3 shows that the ideal $\langle I, x \rangle$ is proper. The weak maximality of I now ensures that $\langle I, x \rangle = I$ and therefore $x \in I$. Hence $\sim\sim I \subset I$ and so $\sim\sim I = I$. ▮

Proposition 2.2.5 *If R is quasidiscrete, then for each maximal ideal I of R, $\sim\sim I$ is a stable maximal ideal.*

Proof Let $x, y \in \sim\sim I$ and $a \in R$. To show that $x - ay \in \sim\sim I$, first consider the case where $x \in I$. Since I is maximal, for each $z \in \sim I$ there exist $p \in I$ and $q \in R$ such that $e = p + qz$. Suppose that $p + q(x - ay)$ is invertible. Then by Lemmas 2.2.1 and 2.2.2, $x - ay \in \sim I$. For each $y' \in I$, since $x - ay' \in I$, we have $x - ay \neq x - ay'$, that is $ay - ay' \neq 0$ hence $a\,(y - y') \neq 0$ and therefore $y \neq y'$. Thus $y \in \sim I$, a contradiction. Thus $p + q(x - ay)$ is not invertible. Since the inequality is quasidiscrete,

$$p + q(x - ay) \neq e = p + qz$$

entails that $x - ay \neq z$. Since $z \in \sim I$ is arbitrary, it follows that $x - ay \in \sim\sim I$. This disposes of the case $x \in I$.

We now turn to the general case. Let z, p, q be as before and consider any $x' \in I$. By the foregoing, $x' - ay \in \sim\sim I$. If $p + q(x - ay)$ is invertible, then by Lemmas 2.2.1 and 2.2.2, $x - ay \in \sim I$ and therefore $x - ay \neq x' - ay$ whence $x \neq x'$. Since $x' \in I$ is arbitrary, this implies that $x \in \sim I$, again a contradiction from which we conclude that $p + q(x - ay)$ is not invertible. Arguing as at the end of the case $x \in I$, we now obtain $x - ay \in \sim\sim I$. Since $x, y \in \sim\sim I$ and $a \in R$ are arbitrary, it follows that $\sim\sim I$ is an ideal in R.

Now, since $e \in \sim I = \sim\sim\sim I$ and

$$\sim\sim (\sim\sim I) = \sim (\sim\sim\sim I) = \sim\sim I,$$

the ideal $\sim\sim I$ is both proper and stable. If $x \in \sim (\sim\sim I)$, then $x \in \sim I$; so $\langle I, x \rangle = R$ and therefore $\langle \sim\sim I, x \rangle = R$. Hence $\sim\sim I$ is maximal. ■

If I is an ideal of R such that $\sim\sim I$ is maximal (respectively, weakly maximal), the most natural question one must ask:

> *Is I itself also a maximal ideal (respectively, weakly maximal) when $\sim\sim I$ is maximal?*

Unfortunately, as in our next Brouwerian example, this is not so constructively even for a discrete ring. Recall that an ideal I in a ring R is ***prime*** if

$$\forall x, y \in R(xy \in I \Rightarrow (x \in I \vee y \in I)).$$

Brouwerian Example 2.2.6 We show that the statement

> *Every proper ideal I of $\mathbb{Z}$ such that $\sim\sim I$ is a maximal (respectively, weakly maximal) ideal is itself maximal (respectively, weakly maximal).*

implies **LEM**.

Let P be any syntactically correct statement such that $\neg\neg P$ holds, and let I be the ideal generated in the ring $\mathbb{Z}$ of the integers by the set

$$G := \{n \in \mathbb{Z} : n = 2 \wedge P\}.$$

Then $\sim\sim I$ is a prime and (weakly) maximal ideal generated by $\langle 2 \rangle$. But if I is maximal, then we can find $m \in I$ and $n \in \mathbb{Z}$ such that $m + 3n = 1$. Since $m \in I$ and, clearly $m \neq 0$, we must have P. On the other hand, being a subset of $\sim\sim I$, the ideal $\langle I, 2 \rangle$ is proper. So if I is weakly maximal, then $2 \in I$ and again we obtain P. ☺

The next three results shed more light on the stability of maximal ideals.

Proposition 2.2.7 *Let I be a maximal ideal of R, and J a proper ideal of R that includes I. Then $\sim I =\sim J$.*

Proof Let $x \in\sim I$. Then there exist $p \in I$ and $q \in R$ such that $p + qx = e \in\sim I$. For each $y \in J$ we have $p + qy \in J$ and so $p + qy \neq e$ since J is a proper ideal; whence $p+qy \neq e = p+qx$, that is $q(x-y) \neq 0$ and therefore $x \neq y$. Thus $x \in\sim J$. Since $x \in\sim I$ is arbitrary, we can conclude that $\sim I \subset\sim J$. The reverse inclusion is straight forward. ∎

Corollary 2.2.8 *Let I be a maximal ideal of R, and J a proper ideal of R that includes I. Then $J \subset\sim\sim I$.*

Proof By Proposition 2.2.7, $J \subset\sim\sim J =\sim\sim I$. ∎

Corollary 2.2.9 *A stable maximal ideal of a ring R is weakly maximal.*

Proof Let I be a stable maximal ideal, and let $x \in\sim\sim I$. Since I is stable, by Corollary 2.2.8, we can say that $x \in I$. That is, $\langle I, x \rangle = I$ and it is a proper ideal. So we conclude that I is a weakly maximal ideal of R. ∎

As an aside, we now consider the question

> *If an ideal I of the ring R is proper, is $\langle \sim\sim I\rangle$ also proper?*

If the ring is discrete, then the answer is "yes". Recall that a proper ideal of a ring R is ***coadditive*** if

$$\forall x, y \in R(x + y \in\sim I \Rightarrow x \in\sim I \vee y \in\sim I).$$

Proposition 2.2.10 *Let R be a quasidiscrete ring and I a proper ideal of R that is coadditive. Then $\langle \sim\sim I\rangle$ is a proper ideal.*

Proof For $1 \leq k \leq n$, let $p_k \in R$ and $x_k \in\sim\sim I$. Either

$$e \neq p_1x_1 + \ldots + p_nx_n \tag{2.1}$$

or else $p_1x_1 + \ldots + p_nx_n$ is invertible. In the latter case, Lemma 2.2.1 shows that $p_1x_1 + \ldots + p_nx_n \in\sim I$; whence, by coadditivity, there exists k such that $p_kx_k \in\sim I$. For each $x \in I$ we then have $p_kx_k \neq p_kx$, which is $p_kx_k - p_kx \neq 0$ so $p_k (x_k - x) \neq 0$ and therefore $x_k \neq x$. Thus $x_k \in\sim I$, a contradiction. We then conclude that (2.1) holds. Since $p_1x_1 + \ldots p_nx_n$ is an arbitrary element of $\langle \sim\sim I\rangle$, it follow that $\langle \sim\sim I\rangle$ is a proper ideal. ▮

Constructively, not every proper ideal of $\mathbb{Z}$ can be prove coadditive. Consider the ideal I generated by

$$\{6\} \cup \{n \in \mathbf{Z} : n = 2 \wedge P\} \cup \{n \in \mathbf{Z} : n = 3 \wedge \neg P\},$$

where P is any syntactically correct mathematical statement: we have $2 + 3 \in\sim I$, but if $2 \in\sim I$, then $\neg P$, while if $3 \in\sim I$, then $\neg\neg P$.

2.3 Weakly maximal and maximal ideals

The main focus in this section is on the questions

Does weakly maximality imply maximality?

Proposition 2.3.1 *If I is a maximal ideal in R, then for all $x \in R$,*

$$x \in\sim I \Rightarrow x^2 \in\sim I. \tag{2.2}$$

Proof Let $x \in\sim I$, there exist $a \in I$ and $b \in R$ such that

$$e = a + bx \tag{2.3}$$

Multiplying (2.3) by x we have $ex = ax + bx^2$. Note that $ex = x$, so we have $x = ax + bx^2$, or we can write this as

$$bx^2 = x - ax. \tag{2.4}$$

For each $y \in I$ we have $ax + by \in I$ and therefore

$$b\left(x^2 - y\right) = bx^2 - by.$$

Replacing bx^2 in this expression by the right hand side of (2.4), we have

$$b\left(x^2 - y\right) = x - ax - by = x - (ax + by) \neq 0$$

and hence $b\left(x^2 - y\right) \neq 0$. Thus $x^2 - y \neq 0$ and therefore $x^2 \neq y$. So $x^2 \in\sim I$ ∎

A proper ideal I of a ring R is ***semiprime*** if

$$\forall x, y \in R((xy \in I \wedge x \in\sim I) \Rightarrow y \in I).$$

Proposition 2.3.2 *Let I be a weakly maximal ideal in R such that* (2.2) *holds. Then I is semiprime.*

Proof Consider $x, y \in R$ such that $xy \in I$ and $x \in\sim I$. Let $M = \langle I, y\rangle$ be an ideal. For each $a \in I$ and $b \in R$, we have

$$x - a - by \neq 0 \tag{2.5}$$

Now, we multiply (2.5) by x, we have

$$x(x-a-by) = x^2 - (ax+bxy) \neq 0.$$

Since $ax+bxy \in I$, by Proposition 2.3.1, $x^2 \in\sim I$. Hence $x-a-by \neq 0$ and therefore $x \neq a+by$. It follows that $x \in\sim M$, so M is a proper ideal that includes I. Since I is weakly maximal, we must have $M = I$ and therefore $y \in I$. Thus I is semiprime. ▮

We call a ring R

- ▷ a ***cancellation domain*** if for all x, y in R,

  $$(xy = 0 \wedge x \neq 0) \Rightarrow y = 0;$$

- ▷ an ***FGP ring*** (Finitely Generated Principle Ideal) if every finitely generated ideal in R is principal;
- ▷ a ***principal ideal ring*** if it is an **FGP** ring and satisfies the ***divisor chain condition***: for each ascending chain $I_1 \subset I_2 \subset \cdots$ of principal ideals in R, there exists n such that $I_n = I_{n+1}$.

As discussed in [8], a principal ideal ring that is also a domain (respectively, cancellation domain) is called ***principal ideal domain*** (respectively, principal ideal cancellation domain). The canonical example of principal ideal domain is the ring $\mathbb{Z}$ of integers. To be specific, we cannot prove constructively that it is a principal ideal domain in the classical sense and for that reason we include the divisor chain condition as part of the definition.

Before we prove our main result, we state the following result which is Proposition 2 on page 2793 of [8]. Recall that a function f from set A to set A is said to be ***strongly extensional*** if $a_1 \neq a_2$ whenever $f(a_1) \neq f(a_2)$.

Proposition 2.3.3 *Let I be an ideal in R, and x, y are elements of R. If $xy \in\sim I$, then $x \in\sim I$ and $y \in\sim I$. Suppose also that addition is strongly extensional and that I is reflective, then I is coadditive.*

Theorem 2.3.4 *If R is a principal ideal cancellation domain, then the following are equivalent conditions on a weakly maximal ideal I.*

(i) *I is maximal.*

(ii) $\forall x \in R\,(x \in\sim I \Rightarrow x^2 \in\sim I)$.

(iii) *I is semiprime.*

Proof Proposition 2.3.1 shows that (i) implies (ii), and Proposition 2.3.2 that (ii) implies (iii). So it remains to show that (iii) implies (i).

Assume that R is a principal ideal domain. Let I be a nonzero semiprime ideal in R, and consider any $x \in\sim I$. For any nonzero $m \in I$ the ideal $\langle m, x\rangle$ is principal, generated by the single element d, say. Pick elements $u, v \in R$ such that $d = ux+vm$. Since d divides x, we see from Proposition 2.3.3 that $d \in\sim I$. Choose $m_1 \in R$ such that $m = m_1 d$. Then I is semiprime and that $m_1 \in I$. Repeating this argument, we construct nonzero elements $m_0 = m, m_1, m_2, \ldots$ of I such that

- $\langle m_0\rangle \subset \langle m_1\rangle \subset \langle m_2\rangle \subset \cdots$, and
- for each k there exist $u_k, v_k, d_k \in R$ such that $u_k x + v_k m_k = d_k$ and $m_k = d_k m_{k+1}$.

By the divisor chain condition, there exist k and an element z of R such that $m_k z = m_{k+1}$. Then, as R is a cancellation domain, we have $e = d_k z = (u_k x + v_k m_k)\, z \in \langle I, x\rangle$ and therefore $\langle I, x\rangle = R$. Thus I is maximal. ∎

Proposition 2.3.5 *If R is quasidiscrete, then every weakly maximal ideal of R is semiprime.*

Proof Let I be a weakly maximal ideal of R, and let $x, y \in R$ such that $xy \in I$ and $x \in\sim I$. For all $a \in I$ and $b \in R$, either $e \neq a + by$ or $a + by$ has an inverse z, since R is quasidiscrete . In the latter case, we can write as $e = z(a + by)$ or

$$e = za + zby. \tag{2.6}$$

We multiply (2.6) by x to get

$$xe = xza + xzby \quad \Rightarrow \quad x = (xz)a + (bz)xy \in I,$$

a contradiction since we let $x \in\sim I$. It now follows that $e \in\sim \langle I, y\rangle$. Since I is weakly maximal, we have $\langle I, y\rangle = I$ and hence $y \in I$. ▮

Theorem 2.3.6 *Every weakly maximal ideal of a quasidiscrete principal ideal cancellation domain is maximal.*

Proof This follows from Proposition 2.3.5 and Theorem 2.3.4. ▮

Corollary 2.3.7 *A weakly maximal ideal in $\mathbb{Z}$ is maximal.*

Proof Since $\mathbb{Z}$ is (quasi)discrete and a principal ideal cancellation domain, Theorem 2.3.6 applies. ▮

Corollary 2.3.8 *The following are equivalent conditions on an ideal I in a quasidiscrete principal ideal cancellation domain.*

(i) *I is weakly maximal.*

(ii) *I is stable and maximal.*

Proof (i)⇒(ii) Let I be a weakly maximal ideal in R where R is quasidiscrete and therefore I is stable. Let $x, y \in R$ such that $xy \in I$ and $x \in\sim I$. By Proposition 2.3.5, $y \in I$ and therefore I is semiprime. It now follows from Theorem 2.3.4 that I is a maximal ideal.

(ii)⇐(i) Suppose I is stable and maximal. Then, by Corollary 2.2.9, I is weakly maximal. ▮

Interestingly, we note that if we consider ideals in a Banach algebra $\mathcal{B}$, then, although we trade in the discrete inequality of $\mathbb{Z}$, we gain as partial compensation the existence of inverses of nonzero elements of $\mathcal{B}$.

We end this section with the question

> *Does the Banach algebra structure enable us to prove that if $I \subset \mathcal{B}$ is an ideal such that $\sim\sim I$ is a maximal ideal, then I is maximal?*

Our next example shows that this is not the case. But first, we state the following result which is due to Bishop ([5], page 92).

Lemma 2.3.9 (Bishop's) *Let A be a complete, located*[2] *subset of a metric space X, and x a point of X. Then there exists a point a in A such that $\rho(x, a) > 0$ entails $\rho(x, A) > 0$.*

Brouwerian Example 2.3.10 We show that assuming **MP** (which is equivalent to[3] $\forall x \in \mathbb{R}(\neg(x = 0) \Rightarrow x \neq 0)$), the statement

> *If I is a proper closed ideal of a Banach algebra such that $\sim\sim I$ is a maximal ideal, then I is maximal.*

implies **LEM**.

Let $\mathcal{B}$ be the Banach algebra $C[0, 1]$ with the usual sup norm, let P be any syntactically correct mathematical statement, and let

$$S \equiv \{f \in \mathcal{B} : f(0) = 0 \wedge (P \vee \neg P)\}.$$

Let I be the (proper) closed ideal of $\mathcal{B}$ generated by the set $\{x^2\} \cup S$, where we write $1, x, x^2, \ldots$ to represent the power functions. Then $f(0) = 0$ for each $f \in \sim\sim I$. On the other hand, for each $g \in \sim I$, if $g(0) = 0$, then $\neg(P \vee \neg P)$, since if $(P \vee \neg P)$, then

---

[2] A subset A of a metric space X is ***located*** in X if the distance $\rho(x, A) \equiv \inf\{\rho(x, y) : y \in A\}$ from x to A exists for every x in X.

[3] Note that in (the metric space) $\mathbb{R}$, $x \neq y$ means that the elements x and y are a positive distance apart.

$g \in I$. But $\neg(P \vee \neg P)$ is absurd; so $\neg(g(0) = 0)$ and therefore, by **MP**, $g(0) \neq 0$. It follows that if $f \in \mathcal{B}$ and $f(0) = 0$, then $f \in \sim\sim I$. Hence

$$\sim\sim I = \{f \in \mathcal{B} : f(0) = 0\},$$

which, as is well known (see [14]), is a maximal ideal of $\mathcal{B}$. Now suppose that I itself is maximal. Then, since $1 + x \in\sim I$,

$$\mathcal{B} = \langle I, 1 + x \rangle. \tag{2.7}$$

Let V be the finite dimensional subspace of $\mathcal{B}$ with basis $\{1, x, x^2\}$. Since, as is easily proved, $x^3 \in\sim V$, we see from Bishop's Lemma that

$$0 < d \equiv \rho\left(x^3, V\right).$$

By (2.7), there exist complex numbers $\zeta_0, \ldots, \zeta_{m+1}$ and elements $f_1, \ldots, f_m$ of S such that

$$\sup_{0 \leq t \leq 1} \left| t^3 - \zeta_0 t^2 - \sum_{k=1}^{m} \zeta_k f_k(t) - \zeta_{m+1}(1 + t) \right| < \frac{1}{2} d.$$

Then

$$\begin{aligned} \left| \sum_{k=1}^{m} \zeta_k f_k \right| &\geq \sup_{0} \left| t^3 - \left(\zeta_{m+1} + \zeta_{m+1} t + \zeta_0 t^2 \right) \right| - \frac{1}{2} d \\ &\geq \rho\left(x^3, V\right) - d = \frac{1}{2} d > 0. \end{aligned}$$

It follows that there exists k such that $\zeta_k f_k \neq 0$. Hence S is inhabited, and therefore $P \vee \neg P$ holds.

Chapter 3

Ideals in a Banach Algebra

In this chapter, we continue the investigation of ideals but this time the focus would be primarily in the context of Banach algebras. As already highlighted earlier, the result and presentation throughout this chapter are adapted from the works by Bridges et al. that appeared in [15]. Firstly, we dispose of some definitions and background information relating to the very notion of a 'quasimaximal ideal' and make an interesting observation with regard to the compactness of the spectrum of a Banach algebra.

3.1 Quasimaximal ideals and compactness of the spectrum

We have seen in Brouwerian Example 1.4.3 that we can't hope to prove the compactness of the spectrum constructively. Write $\mathcal{B}_1^*$ to denote the unit ball of $\mathcal{B}^*$. Let $(r_n)_{n=1}^{\infty}$ be a decreasing sequence of positive numbers converging to 0 and consider the sets

$$\Sigma_{r_n} = \{u \in \mathcal{B}_1^* : |u\,(a_i a_j) - u\,(a_i)\,u\,(a_n)| \leq r_n\,(1 \leq i, j \leq n), |1 - u(e)| \leq r_n\} \quad (3.1)$$

Since $\mathcal{B}_1^*$ is weak* compact[1] and for each $a \in \mathcal{B}$, the mapping $u \mapsto u(a)$ is uniformly continuous on $\mathcal{B}_1^*$, we can compute a strictly decreasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers converging to 0 such that (3.1) is either compact or empty ([5], page 460, Proposition (2.7)). Moreover,

$$\Sigma = \bigcap_{n=1}^{\infty} \Sigma_{r_n}$$

and instead of working directly Σ, we choose to work with Σ_{r_n} which can be seen as an approximations to Σ. This approach was first introduced by Bishop in [2] and again by Bishop and Bridges with a finer and smoother development in the last chapter of [5].

A ***quasimaximal*** ideal I (or ***q-ideal***) of a commutative, unital (complex) Banach algebra $\mathcal{B}$ is a set of the form

$$\left\{ x \in \mathcal{B} : \lim_{n \to \infty} u_n(x) = 0 \right\}$$

where $u_n(x)$ is a sequence of character in Σ_{r_n}.

We introduce the notion of a 'q–ideal' or quasimaximal ideal in a commutative Banach Algebra $\mathcal{B}$, and investigate the relation between the q–ideals and maximal ideals constructively.

As briefly mentioned earlier, the fundamentals of the constructive theory of commutative Banach Algebra were first discussed in the final chapter of [2] with a greater development that appeared in [5]. We begin by stating some of those fundamentals which are proved in chapter 9 of [5].

We now look at an example which shows the noncompactness of $\sum$. Fix integers $r_0 \geq r_1 \geq \ldots$ so that for each n either $r_n = 0$ or $r_n = 1$, and so we are unable to rule out either the possibility that $r_n = 1$ for all n or the possibility that $r_n = 0$ for some n. Let $\mathcal{B}$ consists of all sequences $\mathbf{x} \equiv (x_n)_{n=0}^{\infty}$ of complex numbers for which

$$\|x\| \equiv \sum_{n=0}^{\infty} r_n \, |x_n| \tag{3.2}$$

---

[1] This is compactness in the weak* topology.

exists. Define the elements $\mathbf{x}$ and $\mathbf{y} \equiv (y_n)_{n=0}^{\infty}$ of $\mathcal{B}$ to be equal if $\|x - y\| = 0$. Then $\mathcal{B}$ is a Banach space with norm given by (3.2). Moreover, if we defined the product of $\mathcal{B}$ by

$$\mathbf{xy} \equiv \left(\sum_{k=0}^{n} x_k y_{n-k} \right)_{n=0}^{\infty}.$$

Then $\mathcal{B}$ is a Banach Algebra. In case $r_n = 1$ for all n, every complex number z with $|z| \leq 1$ defines an element u_z of the spectrum Σ of $\mathcal{B}$ by

$$u_z(x) \equiv \sum_{n=0}^{\infty} x_n z^n \qquad (x \in \mathcal{B}).$$

Since neither of these possibilities is ruled out, Σ is not totally bounded, and so is *not* compact.

Bishop‘s main result in Banach Algebra theory is the fundamental tool for solving equations and the next theorem is amongst the key results.

Theorem 3.1.1 *If* $x_1, \ldots, x_n$ *are elements of* $\mathcal{B}$, δ *a positive number, and* N *is a positive integer such that*

$$|u(x_1)| + \cdots + |u(x_n)| \geq \delta \qquad (u \in \Sigma_{r_n}),$$

then there exist $y_1, \ldots, y_n$ *in* $\mathcal{B}$ *such that*

$$x_1 y_1 + \ldots + x_n y_n = e.$$

The proof is long and ingenious and it can be found on pages 453–459 of [5]. However, in the final chapters of [2,5], it is quite clear that Bishop avoided the notion of maximal ideal but rather concentrating instead on the solution of equations of the form

$$\sum_{i=1}^{n} x_i y_i = e$$

in the algebra.

3.2 Locating q–ideal

Recall that an ideal of $\mathcal{B}$ is maximal if and only if it is the kernel of the character of $\mathcal{B}$. Our next considerations are prompted by the observation that, although we cannot expect to prove by purely constructive means that every proper ideal of a Banach Algebra $\mathcal{B}$ with identity is contained in a maximal ideal, there is an explicit classical description of the maximal ideals of $\mathcal{B}$. The natural question one might ask is: Can we produce a constructive substitute for this classical result? But first we need to define some notions which prove central to the discussions to follow.

Given that classically, the maximal ideals $\mathcal{B}$ are precisely the kernels of characters, we introduce the following notion of approximate ideal as a weakening of the notion of 'character'.

By an ***approximate character*** of $\mathcal{B}$, we mean a pair (r, u) consisting of an admissible sequence $r = (r_n)_{n\geq 1}$ and a sequence $u = (u_n)_{n\geq 1}$ such that $u_n \in \Sigma_{r_n}$ for each n. The corresponding q–ideal is

$$M_{r,u} = \left\{ x \in \mathcal{B} : \lim_{n\to\infty} u_n(x) = 0 \right\} \tag{3.3}$$

A subset S of a metric space (X, ρ) is ***located*** if the distance

$$\rho(x, S) := \inf \{\rho(x, s) : s \in S\}$$

exists for each $x \in X$. This is perhaps one of the important properties in constructive analysis though appeared to be classically trivial if not insignificant. Being located is merely saying that we can actually locate (that is, compute the distance to) an object or a set. In particular, we would be very much interested in locating located maximal ideals in $\mathcal{B}$.

Note that a proper ideal of a commutative Banach algebra is located and maximal if and only if the kernel of a (unique) character of the algebra (Theorem 7 of [7], and also refer to Theorem 3.2.3 below).

To prove our next result we need the following result which is proved in [5] (page 303), but first the following definition.

Let u be a bounded linear map of a normed linear space X into a normed linear space Y. We say that u is ***normable*** if

$$\|u\| \equiv \sup\{\|u(x)\| : x \in X, \|x\| = 1\}.$$

Proposition 3.2.1 *A nonzero bounded linear functional u on a normed linear space X is normable if and only if $\ker u$ is located.*

Proposition 3.2.2 *If $u \in \Sigma$, then the $\ker u$ is a closed located maximal ideal of $\mathcal{B}$.*

Proof Clearly, $\ker u$ is a proper ideal as $u(e) = 1$. Since u is normable, by Proposition 3.2.1, $\ker u$ is located. Since u is continuous, $\ker u$ is a closed subset of $\mathcal{B}$. For any $x \in \sim \ker u$, we have $x - u(x)e \in \ker u$ and therefore $x \neq x - u(x)e$. Hence $u(x) \neq 0$, and so

$$e = u(x)^{-1}x - u(x)^{-1}(x - u(x)e) \in (\ker u, x).$$

It now follows that $(\ker u, x) = \mathcal{B}$. ▮

The preceding proposition leads us to the next theorem which is stated without proof. To be precise, the next next result shows that a proper ideal of a commutative Banach algebra is located and maximal if and only if it is the kernel of a (unique) character of the algebra (Theorem 7, page 154 of [7])

Theorem 3.2.3 *The following are equivalent conditions on a proper ideal I of $\mathcal{B}$.*

(i) *I is located and maximal*

(ii) *$I = \ker u$ for some (unique) $u \in \Sigma$*

The next proposition is a Brouwerian example showing that we cannot hope to prove by purely constructive means that every proper ideal of a Banach Algebra with identity is contained in a maximal ideal (Proposition 8, page 154 of [7])

Proposition 3.2.4 *If every proper ideal in $\mathcal{B} = C[0,1]$ is contained in some located maximal ideal, then* **LPO** *holds.*

Proof Let (a_n) be a binary sequence with at most one term equal to 1. Construct a sequence (f_n) of nonnegative elements of $\mathcal{B}$ with the following properties:

- $\triangleright$ If $a_k = 0$ for all $k \leq n$, then $f_n(0) = f_n(1) = \frac{1}{2n}$ and
$$\sup_{0 \leq x \leq 1} |f_n(x) - \sin \pi x| < \frac{1}{n}; \tag{3.4}$$
- $\triangleright$ If $a_n = 1$ and n is even, then $f_n(0) = 0$, $f_n(1) = \frac{1}{2n}$, (3.4) holds, and $f_k = f_n$ for all $k > n$;
- $\triangleright$ If $a_n = 1$ and n is odd, then $f_n(0) = \frac{1}{2n}$, $f_n(1) = 0$, (3.4) holds, and $f_k = f_n$ for all $k > n$.

Then (f_n) is a Cauchy sequence in $\mathcal{B}$: in fact, $\|f_m - f_n\| < \frac{2}{n}$ whenever $m \geq n$. So it converges to a limit $f_\infty \in \mathcal{B}$. Suppose that the principle ideal (f_∞) is contained in a located maximal ideal I of $\mathcal{B}$. Then choose $\xi \in [0,1]$ and let $I = \{f \in \mathcal{B} : f(\xi) = 0\}$. Either $\xi > 0$ or $\xi < 1$. In the first case, $a_n = 0$ for all even n; in the second, $a_n = 1$ for all odd n

Note that the principle ideal (f_∞) of $\mathcal{B}$ is proper. To see this, first note that $\inf(f_\infty) = 0$: for if $\inf(f_\infty) > 0$, then we must have $a_n = 0$ for all n; whence $f_\infty = \sin \pi x$ for all $x \in [0,1]$, and therefore $\inf(f_\infty) = 0$, a contradiction. It now follows that if g is any element of $\mathcal{B}$, then $\inf(gf_\infty) = 0$ and therefore $gf_\infty \neq 1$. Thus $1 \in \sim (f_\infty)$. ∎

Here is an interesting aside on the preceding proof. Define sequence (u_n) in Σ as follows. If $a_k = 0$ for all $k \leq n$, or if $a_n = 1$ for some even $k \leq n$, set $u_n(f) = f(0)$ for all $f \in C[0,1]$. Note that we then have

$$|u_n(f_\infty)| \leq |u_n(f_n)| + \|f_\infty - f_n\| \leq \frac{1}{2n} + \frac{1}{n} = \frac{3}{2n}$$

If $a_k = 1$ for some odd $k \leq n$, set $u_n(f) = f(1)$ for all $f \in C[0,1]$; in this case, $|u_n(f_\infty)| \leq \frac{1}{2n}$. It follows that

$$(f_\infty) \subset \left\{ f \in C[0,1] : \lim_{n\to\infty} u_n(f) = 0 \right\}.$$

The larger set on the right hand side is an example of a quasimaximal ideal (or q–ideal). There are indications that q–ideals may prove more useful than maximal ones in constructive analysis. For example, it is shown in [12] that a q–ideal is a proper closed ideal, and that every separable proper ideal in $\mathcal{B}$ is contained in a q–ideal which is shown in Proposition (3.2.8) below.

Proposition 3.2.5 *The intersection of a countable family of located q–ideal of $\mathcal{B}$ is a q–ideal.*

Proof Let $(M_n)_{n\geq 1}$ be a sequence of located q–ideal. Then for each n there exists a character v_n of $\mathcal{B}$ such that $M_n = \ker v_n$. Let $u = (u_n)_{n\geq 1}$ be the sequence

$$(v_1, v_2, v_1, v_2, v_3, v_1, v_2, v_3, v_4, \ldots)$$

and let $(r_n)_{n\geq 1}$ be any admissible sequence of positive numbers. Then

$$\begin{aligned} x \in \bigcap_{n\geq 1} M_n \quad &\Leftrightarrow \quad \forall n\,(v_n(x) = 0) \\ &\Leftrightarrow \quad \lim_{n\to\infty} u_n(x) = 0. \end{aligned}$$

So, $\bigcap_{n\geq 1} M_n = M_{r,u}$. ∎

Proposition 3.2.6 *If (r, u) is an approximate character of $\mathcal{B}$, then $M_{r,u}$ is a proper closed ideal in $\mathcal{B}$, and $e \in -M_{r,u}$.*

Proof It is clear that $M_{r,u}$ is an additive subgroup of $\mathcal{B}$. Let $x \in M_{r,u}$, $y \in \mathcal{B}$, and $0 < \epsilon < 1$. Choose i, j so large that

$$\|x - a_i\| < (1 + \|y\|)^{-1}\epsilon, \quad \|y - a_j\| < (1 + \|x\|)^{-1}\epsilon, \quad \text{and} \quad \|xy - a_i a_j\| < \epsilon.$$

Then choose n so large that $r_n < \epsilon$, $|u_n(x)| < 1$, $|u_n(y)| < \epsilon$ and $i, j \leq n$. We have

$$\begin{aligned}
|u_n(xy)| &\leq |u_n(xy) - u_n(x)u_n(y)| + |u_n(x)|\,|u_n(y)| \\
&\leq |u_n(xy) - u_n(a_i a_j)| + |u_n(a_i a_j) - u_n(a_i)\,u_n(a_j)| \\
&\qquad + |u_n(a_i)|\,|u_n(a_j) - u_n(y)| + |u_n(y)|\,|u_n(a_i) - u_n(x)| + \epsilon \\
&\leq \|xy - a_i a_j\| + r_n + \|a_i\|\,\|a_j - y\| + \|y\|\,\|a_i - x\| + \epsilon \\
&< \epsilon + \epsilon + (1 + \|x\|)\,\|a_j - y\| + \epsilon + \epsilon \\
&< 5\epsilon.
\end{aligned}$$

Since ϵ is arbitrary, it follows that $xy \in M_{r,u}$. Thus $M_{r,u}$ is an ideal in $\mathcal{B}$.

If $x \in M_{r,u}$, then for each $\epsilon > 0$ and for all sufficiently large n we have $|u_n(x)| < \epsilon < 1 - \epsilon < |u_n(e)|$, so $\|e - x\| > 1 - 2\epsilon$. Hence, $\|e - x\| \geq 1$ and $e \in -M_{r,u}$. It remains to prove that $M_{r,u}$ is closed. To this end, given $x \in \overline{M}_{r,u}$ and $\epsilon > 0$, we choose $y \in M_{r,u}$ such that $\|x - y\| < \epsilon$, and then N such that $|u_n(y)| < \epsilon$ for all $n \geq N$. For all such n, we have

$$|u_n(x)| \leq \|x - y\| + |u_n(y)| < 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{n\to\infty} u_n(x) = 0$ and hence that $x \in M_{r,u}$

∎

Although, as we shall see later, we cannot guarantee that a separable proper ideal is contained in a maximal ideal, we can be sure that it is contained in a q–ideal.

To prove the next result, we state the following which is proved in [5] (page 94).

Proposition 3.2.7 *Let $f : X \to \mathbb{R}$ be an uniformly continuous function on a totally bounded metric space X. Then the supremum and the infimum of f exist.*

Proposition 3.2.8 *Every separable proper ideal in $\mathcal{B}$ is contained in a q–ideal. More exactly, if $(x_n)_{n\geq 1}$ is a dense sequence in a separable proper ideal I of $\mathcal{B}$, and if $r = (r_n)_{n\geq 1}$ is an admissible sequence of positive numbers strictly decreasing to 0,* *then there exists an approximate character* (r, u) *such that* $|u_n(x_k)| < 2^{-n}$ *whenever* $1 \leq k \leq n$, *and such that* $I \subset M_{r,u}$.

Proof For each n, since the mapping

$$u \rightsquigarrow |u(x_1)| + \cdots + |u(x_n)|$$

is uniformly continuous on weak*–compact subset Σ_{r_n} of $\mathcal{B}_1^*$, and by Proposition 3.2.7 the real number

$$\delta_n = \inf\{|u(x_1)| + \cdots + |u(x_n)| : u \in \Sigma_{r_n}\}$$

exists. If $\delta_n > 0$, then there exists $y_1, \ldots, y_n$ in $\mathcal{B}$ such that

$$x_1 y_1 + \ldots + x_n y_n = e,$$

so $e \in I$ and therefore $I = \mathcal{B}$, a contradiction. Hence $\delta_n = 0$. We can therefore construct $u_n \in \Sigma_{r_n}$ such that

$$|u_n(x_1)| + \cdots + |u_n(x_n)| < 2^{-n}.$$

Setting $u = (u_n)_{n \geq 1}$, we see that (r, u) is an approximate character of $\mathcal{B}$. Also, $\lim\limits_{n \to \infty} u_n(x_k) = 0$ for each k. An approximation argument now show that $\lim\limits_{n \to \infty} u_n(x) = 0$ for each $x \in I$. Thus $I \subset M_{r,u}$. $\blacksquare$

Recall that an ideal M in $\mathcal{B}$ is *weakly maximal* if each proper ideal I that contains M actually equals M. The relationship between maximal and weakly maximal ideals is not as simple constructively as it is classically and this has been discussed thoroughly by Bridges and Havea in [14]. Following their treatment, for a single maximal ideal we can trade locatedness in Proposition 3.2.5 for separability plus weak maximality.

Proposition 3.2.9 *A separable weakly maximal ideal* M *in* $\mathcal{B}$ *is a q–ideal. Moreover, if* (r, u) *is an approximate character of* $\mathcal{B}$ *such that* $M = M_{r,u}$ *and if* $u_n(x)$

is bounded away from 0 *for all sufficiently large* n*, then* $x \in -M$*. If also* M *is maximal, then* $x \in\sim M$ *if and only if* $(u_n(x))_{n\geq 1}$ *is bounded away from* 0 *for all sufficiently large* n*, and* $\sim M = -M$*.*

Proof Let $r = (r_n)_{n\geq 1}$ be a strictly decreasing admissible sequence of positive numbers converging to 0. Then, by Proposition 3.2.8, there exists an approximate character (r, u) of $\mathcal{B}$ such that $M \subset M_{r,u}$. Since $M_{r,u}$ is a proper ideal, by Proposition 3.2.6, and M is weakly maximal, we must have $M = M_{r,u}$. Suppose that $|u_n(x)| \geq \delta > 0$ for all $n \geq N_1$. Given $y \in M_{r,u}$, we choose $N \geq N_1$ such that $|u_n(y)| < \frac{\delta}{2}$ for all $n \geq N$. Then for all such n we have

$$\|x - y\| \geq |u_n(x) - u_n(y)| \geq |u_n(x) - u_n(y)| > \frac{\delta}{2}.$$

Hence $x \in -M_{r,u} = -M$.

Now suppose, in addition, that M is maximal, and consider any $x \in\sim M$. Since $(M, x) = \mathcal{B}$, there exist $m \in M$ and $b \in \mathcal{B}$ such that $e = m + bx$. For all sufficiently large n we have

- $|u_n(e) - u_n(m) - u_n(bx)| < \frac{1}{5}$,
- $|u_n(e)| > \frac{4}{5}$, and
- $|u_n(bx) - u_n(b)u_n(x)| < \frac{1}{5}$;

whence

$$\begin{aligned}
|u_n(b)|\,|u_n(x)| &> |u_n(bx)| - \frac{1}{5} \\
&> |u_n(e) - u_n(m)| - \frac{1}{5} - \frac{1}{5} \\
&> |u_n(e)| - |u_n(m)| - \frac{2}{5} \\
&> \frac{4}{5} - |u_n(m)| - \frac{2}{5} \\
&> \frac{2}{5} - |u_n(m)|\,.
\end{aligned}$$

But $u_n(m) \to 0$ as $n \to \infty$, so for sufficiently large n we have $|u_n(m)| < \frac{1}{5}$; whence

$$\frac{1}{5} < |u_n(b)|\,|u_n(x)| \leq \|b\|\,|u_n(x)|$$

and therefore $|u_n(x)| > \frac{1}{5\|b\|}$. It follows from an earlier part of the proof that $x \in -M$. Hence $\sim M \subset -M$. The reverse inclusion is trivial. ▮

One might reasonably hope, or perhaps argue, that q–ideals would be located but in recursive constructive mathematics[2] we can produce an explicit counterexample which shows that it does not happen. To do this, we need the following result which is proved in [5] (pages 381–382).

Proposition 3.2.10 *Let X be a compact space, and $\|\cdot\|'$ an algebra seminorm on $C(X, \mathbb{F})$ such that $\|f\|' \leq \|f\|$ for all $f \in C(X, \mathbb{F})$. Then there exists a compact set $K \subset X$ such that $\|f\|' = \|f\|_K$ for all $f \in C(X, \mathbb{F})$.*

Now, let $\mathcal{B}$ be the Banach algebra $C(I)$ of continuous real–valued functions on $I = [0, 1]$, and let (r_n) be a strictly increasing Specker sequence[3] in I. For each n let u_n be the character of $\mathcal{B}$ defined by $u_n(f) = f(r_n)$. Then

$$M_{r,u} = \left\{ f \in \mathcal{B} : \lim_{n \to \infty} f(r_n) = 0 \right\}$$

is a q–ideal in $\mathcal{B}$. Suppose that $M_{r,u}$ is located. Applying Proposition 3.2.10 to the algebra seminorm $f \rightsquigarrow \rho(f, M_{r,u})$ on $\mathcal{B}$, we see that there exists a compact subset K of I such that

$$\rho(f, M_{r,u}) = \|f\|_K = \sup\{|f(t)| : t \in K\}$$

for each $f \in \mathcal{B}$. In particular,

- ▷ $\lim_{n \to \infty} f(r_n) = 0$ if and only if $\|f\|_K = 0$;

---

[2] This is mathematics with intuitionistic logic plus the Church-Markov-Turing thesis. For further details, see Chapter 3 of [16].

[3] A ***Specker sequence*** is a sequence that is eventually bounded away from each real number.

$\triangleright$ $\rho\left(f, M_{r,u}\right) > 0$ if and only if $\|f\|_K > 0$.

Fixing $x \in I$, choose $\delta > 0$ and a positive integer N such that $|r_n - x| \geq \delta$ for all $n \geq N$. Define $f \in \mathcal{B}$ by

$$f(t) = \max\left\{0, 1 - \delta^{-1}|t - x|\right\}.$$

Then $f\left(r_n\right) = 0$ for all $n \geq N$, so $f \in M_{r,u}$ and therefore $\|f\|_K = 0$. Since $f(x) = 1$, the continuity of f ensures that $\rho(x, K) > 0$. Since $x \in I$ was arbitrary, K must be empty, a contradiction since we assume that K is compact. Therefore, we can conclude that $M_{r,u}$ can not be located.

Is every proper located ideal in a commutative Banach algebra necessarily a q–ideal? We do not know the answer to this question, but in view of the next result and the work of Takamura [28], the answer for C^*–algebras ideals is 'YES'.

Proposition 3.2.11 *Let X be a compact metric space, and I a proper located ideal of $C(X)$. Then I is a q–ideal.*

Proof Define a seminorm on $C(X)$ by

$$\|f\|' := \rho(f, I).$$

Then $\|1\|' > 0$ and $\|fg\| \leq \|f\|\|g\|$ for all $f, g \in C(X)$. It follow from Proposition (3.2.10) that there exists a compact set $K \subset X$ such that

$$\|f\|' = \|f\|_K := \sup\left\{|f(x)| : x \in K\right\}$$

for each $f \in C(X)$. Clearly,

$$f \in I \Leftrightarrow \forall x \in K(f(x) = 0).$$

Construct a dense sequence $\left(x_n\right)_{n\geq1}$ in K. For each n let v_n be the evaluation functional $f \rightsquigarrow f\left(x_n\right)$ on $C(X)$. Then v_n is a character of $C(X)$. Let $u := \left(u_n\right)_{n\geq1}$ be a sequence

$$\left(v_1, v_2, v_1, v_2, v_3, v_1, v_2, v_3, v_4, \ldots\right),$$

and let r be any admissible sequence of positive numbers. Then

$$\begin{aligned} f \in M_{r,u} &\Leftrightarrow \forall n\,(v_n(f) = 0) \\ &\Leftrightarrow \forall n\,(f\,(x_n) = 0) \\ &\Leftrightarrow \forall x \in K(f(x) = 0), \end{aligned}$$

where the last step follows because (x_n) is dense in K and f is uniformly continuous. Thus I is the q–ideal $M_{r,u}$ ∎

Classically, every proper ideal of a commutative Banach algebra is contained in a maximal ideal. If we require the containing maximal ideal to be located (which classically it is, trivially), then, in general, we cannot expect to construct it. This is shown in the following proposition, whose conclusion is much stronger than Proposition (3.2.4). But first we need the next result which is proved in [5] (page 382).

Proposition 3.2.12 *Let K be a compact space, and Γ the set of all nonzero bounded multiplicative linear functionals on $C(K, \mathbb{F})$. For each x in K define the element u_x of Γ by*

$$u_x(f) \equiv f(x) \qquad (f \in C(K, \mathbb{F})).$$

Then Γ is compact in the metric induced by the double norm[4] *on $C(K, \mathbb{F})$, and the map $x \mapsto u_x$ is a metric equivalence of K and Γ.*

Proposition 3.2.13 *If every nonzero proper ideal of a Banach algebra $C[0, 1]$ is contained in a located maximal ideal, then* **WLEM**, *$\neg P \vee \neg\neg P$, is provable.*

---

[4] The notion of a *double norm* was first introduced by Bishop (see page 350 of [5]) under the discussions of dual spaces. To be specific, let X and Y be normed spaces, with X separable. For each dense sequence $(x_n)_{n\geq 1}$ in X, define the corresponding ***double norm*** on the quasinormed space $\mathrm{Hom}(X, Y)$ (that is, bounded linear maps of X into Y) by

$$|||u||| = \sum_{n=1}^{\infty} 2^{-n}\,(1 + \|x_n\|)^{-1}\,\|u\,(x_n)\| \qquad (u \in \mathrm{Hom}(X, Y)).$$

Proof Let $\mathcal{B} = C[0,1]$, let P be any syntactically correct proposition, and define

$$G := \{g \in \mathcal{B} : g(0) = 0 \wedge P\} \cup \{g \in \mathcal{B} : g(1) = 0 \wedge \neg P\} \cup \{g \in \mathcal{B} : g(0) = 0 = g(1)\}\,.$$

Let I be the ideal of $\mathcal{B}$ generated by G. Then I is nonzero, since it contains the function $g \in \mathcal{B}$ such that $g(0) = 0 = g(1)$, $g\left(\frac{1}{2}\right) = 1$, and g is linear on each of the intervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$. To see that I is proper, let $g_1, \ldots, g_n$ be elements of G and $h_1, \ldots, h_n$ elements of $\mathcal{B}$. For each i we have $g_i(0) = 0$ and P, or $g_i(1) = 0$ and $\neg P$, or $g_1(0) = 0 = g_i(1)$. If the first alternative holds for some i, then it holds for all i, so $\sum_{i=1}^{n} g_i h_i(0) = 0$; if $g_i(1) = 0$ and $\neg P$ for some i, then $\sum_{i=1}^{n} g_i h_i(1) = 0$. If neither of these alternatives holds, then we must have $g_i(0) = 0 = g_i(1)$ for each i, so $\sum_{i=1}^{n} g_i h_i(0) = 0$. In each case we have $\sum_{i=1}^{n} g_i h_i \neq 1$. Hence $1 \in\sim I$ and I is proper.

Now, suppose that I is contained in some located maximal ideal M of $\mathcal{B}$. There exists an element $v \in \Sigma$ such that $M = \ker v$. By Proposition 3.2.12, there exists $\xi \in [0,1]$ such that

$$M = \{f \in \mathcal{B} : f(\xi) = 0\}\,. \tag{3.5}$$

Either $\xi > 0$ or $\xi < 1$. In the first case we have $\neg P$; whereas in the second we have $\neg\neg P$. ∎

Note that the ideal I in the preceding proof is not located. Letting ι be the identity function on [0,1], we see that if P holds, then $\iota \in I$; whereas if $\neg P$ holds, then $\rho(\iota, I) = 1$.

In recursive constructive mathematics, we can actually produce an example of a nonzero proper ideal that cannot be contained in a located maximal ideal. In particular, following Bridges et al. in [15], assuming the Church–Markov–Turing thesis, we construct[5] a uniformly continuous mapping f of $[0,1]$ onto $(0,1]$. Let I be the ideal of $C[0,1]$ generated by f, and suppose that I is contained in a located

---

[5] Such construction is demonstrated by Bridges and Richman in Chapter 6 of [**?**].

maximal ideal M. Then there exists a character u of $C[0,1]$ such that $M = \ker u$. By Proposition 3.2.12, there exists $\xi \in [0,1]$ such that (3.5) holds with $\mathcal{B} = C[0,1]$. In particular, $f(\xi) = 0$, which contradicts our construction of f.

Of special interest, the ideal in the preceding paragraph has some other amusing properties. For example, it can be shown (see [15]) that every element of its closure has the form gf with g pointwise continuous on $[0,1]$. Lets look at this more closely. Saying that I is closed is equivalent to saying that it consists of all elements gf with $g \in C[0,1]$. Following the work of Bridges on page 157 of [7], $\sqrt{|f|}$ is in the closure[6] of I. If I is closed, then there exists $g \in C[0,1]$ with $\sqrt{|f|} = gf$. Choosing $M > \|g\|$, for each $x \in [0,1]$ we have either $|f(x)| > \frac{1}{2M^2}$ or $|f(x)| < \frac{1}{M^2}$. In the latter case, if $f(x) \neq 0$, then

$$g(x)^2 = \frac{1}{|f(x)|} > M^2$$

a contradiction; whence $f(x) = 0$. Thus $[0,1]$ is the union of the disjoint closed subsets A and B, where

$$A = \{x \in [0,1] : f(x) = 0\} \quad \text{and} \quad B = \left\{x \in [0,1] : |f(x)| \geq \frac{1}{2M^2}\right\}.$$

Moreover, since f is uniformly continuous on $[0,1]$, there exists $\delta > 0$ such that $\rho(x,y) \geq \delta$ for all $x \in A$ and $y \in B$. It follows that the function h defined in $[0,1]$ by

$$h(x) = \begin{cases} 0 & \text{if } x \in A, \\ \dfrac{1}{f(x)} & \text{if } x \in B \end{cases}$$

is uniformly continuous, and that the principal ideal (f) is generated by the idempotent hf. In particular, since $[0,1]$ is connected, then either A or B is empty; so f and (f) would be 0, which is absurd.

---

[6] This follows from the Weierstraß Approximation Theorem as it appears on page 109 of [5].

3.3 When is a located q–ideal maximal?

We have shown in the last section that a separable proper ideal is contained in q–ideal instead of a maximal ideal. But the question is, when is a located q-ideal maximal? Certainly not always, as Proposition 3.2.11 shows. To further investigate the possible answers to this question and under what conditions, we first state some essential definitions and concepts.

A ***spread*** consists essentially of a tree of finite sequences of natural numbers, such that every sequence has at least one successor, plus a law $\mathcal{L}$ assigning objects of a previously constructed domain to the nodes of the tree.[7]

In the discussion to follow, we shall use spread only for trees of finite sequences of natural numbers without finite branches. The notion of spread is supplemented by the notion of the 'species', much closer to the classical concept of set. In simple terms, one can think of a species as a set of elements singled out from a previously constructed totality by a property (as in separation axiom of classical set theory).

A ***choice sequence*** α of natural numbers is an unfinished, ongoing process of choosing values $\alpha 0, \alpha 1, \alpha 2, \ldots$ by the 'ideal mathematician'[8]. At any stage of his activity the ideal mathematician has determined only finitely many values plus, possibly, some restrictions on future choices.

Brouwer's Continuity Principle: *If to every choice sequence α of a spread, a number $n(\alpha)$ is assigned, $n(\alpha)$ depends on an initial segment $\bar{\alpha}m = \alpha 0, \alpha 1, \ldots, \alpha(m-1)$ only, that is to say for all choice sequences β starting with the same initial segment $\bar{\alpha}m, n(\beta) = n(\alpha)$.*

This principle is not specially singled out by Brouwer, but used in many proofs including the proof of what later known the Bar Theorem. And from the Bar Theorem, Brouwer the following theorem which is very central to intuitionistic mathematics

---

[7] This is a simplified variation of Brouwer's original definition.

[8] This is related to Brouwer's view that mathematics ia a free-creation of the human mind.

in so many ways.

Theorem 3.3.1 (Fan Theorem) *If to every choice sequence α of a finitely branching spread (fan) a number $n(\alpha)$ is assigned, there is a number m, such that for all $\alpha, n(\alpha)$ may be determined from the first m values of α (that it, an initial segment of length m).*

The Fan Theorem may be seen as a combination of the compactness of finite trees with the continuity principle. Note that its classical contrapositive form is Konig's Lemma[9].

The next result (whose proof can be found on page 125 of [16]) is a special representation of certain compact intervals in $\mathbb{R}$ using the fan of all finite sequences in $\{-1, 1\}$, denoted by C, where the first index is 1.

Lemma 3.3.2 *If $\frac{1}{2} < \theta < 1$, then for each infinite path α in C, the series $\sum_{i=1}^{\infty} \alpha_i \theta^i$ converges to a point of the closed interval $I \equiv \left[\frac{-\theta}{1-\theta}, \frac{\theta}{1-\theta}\right]$; and each point of I has the form $\sum_{i=1}^{\infty} \alpha_i \theta^i$ for some path α in C.*

Let X be a normed linear space, and suppose that $f_n, f \in X^*$. Then we say that f_n ***converges weak**** to f, and write $f_n \xrightarrow{w^*} f$, if for all $x \in X$

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Note that weak* convergent is just pointwise convergence of f_n.

Now, answering our question above, we need to work within the framework of intuitionistic mathematics; that is, we add both Brouwer's Continuity Principle (**CP**) and his Fan Theorem for detachable bars ($\mathbf{FT}_D$) to our intuitionistic logic. Following Bridges et al. [15], we begin with a general intuitionistic result. In the follow, write "$A \vdash B$" to mean that "A proves B".

---

[9] Konig's Lemma: If there is no finite upper bound to the lengths of paths in a finitary tree, then there is at least one infinite path in the tree.

Proposition 3.3.3 ***CP***+***FT***$_D$ ⊢ *If* $(z_n)_{n\geq 1}$ *is a sequence of complex numbers that is eventually bounded away from each element of the closed unit disc, then there exists a positive integer* ν *such that* $|z_n| \geq 1 + 2^{-\nu}$ *for all* $n \geq \nu$*.*

Proof Fix θ with $\frac{1}{2} < \theta < 1$, and let $I = \left[\frac{-\theta}{1-\theta}, \frac{\theta}{1-\theta}\right]$. It suffices to prove that if $(z_n)_{n\geq 1}$ is a sequence in $\mathbb{C}$ that is eventually bounded away from each point of the square $I \times I$, then there exists a positive integer ν such that $\rho(z_n, I \times I) \geq 2^{-\nu}$ for all $n \geq \nu$. Let X be the fan consisting of all infinite sequences in $\{-1, 1\}$. According to Lemma 3.3.2, the function

$$\mathbf{x} \rightsquigarrow \sum_{k=1}^{\infty} x_k \theta^k$$

maps X onto I. Thus the function $F : X \times X \to \mathbb{C}$ defined by

$$F(\mathbf{x}, \mathbf{y}) := \sum_{k=1}^{\infty} x_k \theta^k + \mathrm{i} \sum_{k=1}^{\infty} y_k \theta^k$$

maps $X \times X$ onto $I \times I$. Now, X is compact with respect to the metric

$$\rho(\mathbf{x}, \mathbf{y}) := \min \left\{ n \in \mathbb{N}^+ : \overline{\mathbf{x}}(n) = \overline{\mathbf{y}}(n) \right\},$$

so $X \times X$ is compact with respect to the corresponding product metric. Let $(z_n)_{n\geq 1}$ be a sequence in $\mathbb{C}$ that is eventually bounded away from each point of $I \times I$. Then for each $(\mathbf{x}, \mathbf{y}) \in X \times X$ there exists a positive integer N such that

$$|z_n - F(\mathbf{x}, \mathbf{y})| > 2^N \qquad (n \geq N).$$

By **CP**, there exists a continuous function $f : X \times X \to \mathbb{N}^+$ such that

$$|z_n - F(\mathbf{x}, \mathbf{y})| > 2^{-f(\mathtt{x},\mathtt{y})} \qquad ((\mathtt{x}, \mathtt{y}) \in X \times X, n \geq f(\mathtt{x}, \mathtt{y})).$$

The function

$$g : (\mathtt{x}, \mathtt{y}) \rightsquigarrow 2^{-f(\mathtt{x},\mathtt{y})}$$

is then continuous on $X \times X$. Since $X \times X$ is compact, a consequence (actually, an equivalent) of **FT**$_D$ shows that

$$\inf \{ g(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in X \times X \} > 0.$$

Choosing a positive integer ν such that $2^{-\nu}$ is less than this infimum, we see that

$$|z_n - F(\mathbf{x}, \mathbf{y})| > 2^{-\nu} \qquad ((\mathbf{x}, \mathbf{y}) \in X \times X, n \geq \nu).$$

Since F maps $X \times X$ onto $I \times I$, it follows that $\rho\left(z_n, I \times I\right) \geq 2^{-\nu}$ for all $n \geq \nu$. ■

Proposition 3.3.4 ***CP*** + ***FT***$_D$ $\vdash$ *Let (r, u) be an approximate character of the commutative Banach algebra $\mathcal{B}$, such that $M_{r,u}$ is located and has the property that for each $x \in -M_{r,u}$ the sequence $(u_n(x))_{n \geq 1}$ is eventually bounded away from 0. Then $M_{r,u}$ is maximal.*

Proof Writing $M := M_{r,u}$, we see from Proposition 3.2.6 that $\|e\|_{B/M} = \rho(e, M) > 0$. Hence $\mathbb{C}e$ is a 1–dimensional, and therefore located, subspace of the quotient space B/M. Given $x \in -M$, suppose that $\rho_{B/M}(x, \mathbb{C}e) > 0$. Then for each $\lambda \in \mathbb{C}$ we have $x - \lambda e \in -M$; whence there exist a positive integer N and a positive number δ such that $|u_n(x - \lambda e)| > 2\delta$ for all $n \in N$. We may assume that for all such n we also have $|1 - u_n(e)| < \frac{\delta}{1+|\lambda|}$ and therefore

$$\begin{aligned} |u_n(x) - \lambda| &\geq |u_n(x - \lambda e)| - |u_n(\lambda e) - \lambda| \\ &> 2\delta - |\lambda| \, |u_n(e) - 1| \\ &> \delta. \end{aligned}$$

Thus the sequence $(u_n(x))_{n \geq 1}$ is eventually bounded away from each element of $\mathbb{C}$. In particular, it is eventually bounded away from each element of the disc $\overline{B}(0, \|x\|)$ in $\mathbb{C}$. It follows from Proposition 3.3.3 that there exists a positive integer ν such that $|u_n(x)| \geq \|x\| + 2^{-\nu}$ for all $n \geq \nu$. This is absurd, since $|u_n(x)| \leq \|x\|$ for each n. We conclude that $\rho_{B/M}(x, \mathbb{C}e) = 0$. Since $\mathbb{C}e$, being finite–dimensional, is closed in B/M, there exists $\zeta \in \mathbb{C}$ such that $x =_{B/M} \zeta e$ and therefore $x - \zeta e \in M$. It follows that

$$\|\zeta e\| = \|x - (x - \zeta e)\| \geq \rho(x, M) > 0$$

so $\zeta \neq 0$ and therefore $e = \zeta^{-1}(x - (x - \zeta e)) \in (M, x)$. Hence $(M, x) = B$. $\blacksquare$

In the next result, note that it does not require either **CP** or $\mathbf{FT}_D$.

Proposition 3.3.5 *Let $M_{r,u}$ be a q–ideal. Then $M_{r,u}$ is located and maximal if and only if the sequence $(u_n)_{n\geq 1}$ is weak*–convergent in $\mathcal{B}^*$ the dual of $\mathcal{B}$.*

Proof Suppose that $M_{r,u}$ is located and maximal. Then there exists a character v of $\mathcal{B}$ such that $M_{r,u} = \ker v$. For each $x \in \mathcal{B}$ we have $x - v(x)e \in \ker v$, so

$$0 = \lim_{n\to\infty} u_n(x - v(x)e) = \lim_{n\to\infty} (u_n(x) - v(x)u_n(e)) = \lim_{n\to\infty} u_n(x) - v(x),$$

since $u_n(e) \to 1$ Hence $(u_n)_{n\geq 1}$ is weak*–convergent to v.

Now suppose, conversely, that $(u_n)_{n\geq 1}$ is weak*–convergent to an element v of $\mathcal{B}^*$. Then for all i, j we have

$$\begin{aligned} \left|v\left(a_i a_j\right) - v\left(a_i\right) v\left(a_j\right)\right| &= \lim_{n\to\infty} \left|u_n\left(a_i a_j\right) - u_n\left(a_i\right) u_n\left(a_j\right)\right| \\ &\leq \lim_{n\to\infty} r_n \\ &= 0. \end{aligned}$$

Since $(a_n)_{n\geq 1}$ is dense in $\mathcal{B}$, it follows that v is a character. It is simple to prove that $M_{r,u} = \ker v$; whence $M_{r,u}$ is located and maximal. $\blacksquare$

Chapter 4

Conclusion

In this short though thorough investigation, we have seen how applicable and challenging the constructive approach particularly Bishop style to classical Banach algebra theory specifically in the context ideals.

Working constructively—that is, doing mathematics using intuitionistic logic—demands one to appreciate the distinction between idealistic existence (which is synonymous to the impossibility of non–existence) and constructive existence (which is identified as being able to construct or find the object under consideration). The distinction is one that should be heeded and appreciated far more than it is and as Bishop wrote:

> "*Meaningful distinctions deserve to be maintained.*" [5]

It is very clear that the main theme of doing mathematics constructively it to see how does one can find constructive substitutions of already known 'nonconstructive' result while maintaining the classical spirit. As in classical mathematics, maximal and weakly maximal ideals are equivalent. However, in Bishop's constructive mathematics, we have considered the treatment of the stability properties of maximal and weakly maximal ideal in a unital commutative Banach algebra. Some conditions were necessary for a weak maximal ideal to become maximal. However, the various questions raised about maximal ideal being weakly maximal were primarily due to the work of Bridges and Havea in [14].

Brouwerian examples were used in the discussions mainly as a tools to show that some of the classical results about ideal can't carried over into the constructive setting unless some conditions are to added or dropped from the premises.

The work of Bridges et al. in [15] is amongst the current and most recent development in this area and the notion of q–ideal in a commutative Banach algebra is introduced to handle questions regarding maximality of ideals. One of the key questions raised in this paper was how much we can hope for if we want to prove constructively that every proper ideal is contained in a maximal ideal. As we have seen in Chapter 3, it is not simple to prove by purely constructive means, that is using the intuitionistic logic alone. Furthermore, we have seen that that a separable proper ideal is contained in a q–ideal instead of maximal ideal which is perhaps the best constructive substitute by far. There are indications that q–ideals may prove more useful than maximal ones in constructive mathematics. Some examples in [12] show that a q–ideal is a closed ideal, and that every separable proper ideal in $\mathcal{B}$ is contained in q–ideal as already mentioned earlier.

In Chapter 3, we saw that the relationship between maximal and weakly maximal ideals is not as simple constructively as it is classically. For a single maximal ideal we can trade the locatedness for separability plus weak maximality, but the locatedness of a q–ideal is not simple constructively. Though a that q–ideal is not located, we still don't know whether every proper located ideal in a commutative Banach algebra necessarily a q–ideal. However, as shown in Proposition 3.2.11, I is a located q–ideal if I is a proper located ideal of $C(X)$, where X is a compact metric space. The last part of Chapter 3 involved a discussion that directly related to when is a q–ideal maximal and we saw that certainly not always as declared by Proposition 3.2.11. Interestingly enough, working within the framework of intuitionistic mathematics, if **CP** and $\mathbf{FAN}_D$ were assumed, then we would find a solution to this question and this was clearly shown by Proposition 3.3.5.

The approach used in this research to analyzed constructively the context of Banach algebra theory was more straightforward, since it is eliminate and redevelop most of the classical results that were considered by the classical mathematicians. Also we have extended the studies of Banach algebra in constructively and considering more general cases.

Furthermore, numerous attempts were also made to analyze the ideal in constructive Banach algebra theory. As raised by Bridges and Havea particularly in [14], we still these two interesting questions.

1. Is every maximal ideal located?
2. Is every ideal a q–ideal?

Lastly, the research results gathered in this short investigation thus far can hardly be considered to be exhaustive. Rather, the topic belongs to a frontier research field which will be further explored in the future. To what extent these results can be put to any practical purpose will become clearer as studies in this field continue.

Appendix A

Intuitionistic Logic

Working with a fixed first–order language $\mathcal{L}$, we adopt the primitive connectives $\vee$ (or),$\wedge$ (and),$\Rightarrow$ (implies), and $\neg$ (not). In presenting the following axioms, I assume familiarity with basic notions of elementary classical logic; details of these notions may be found in, for example, [1]

Proposition Axiom

1. $p \Rightarrow (p \wedge p)$
2. $(p \wedge q) \Rightarrow (q \wedge p)$
3. $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$
4. $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$
5. $q \Rightarrow (p \Rightarrow q)$
6. $(p \wedge (p \Rightarrow q)) \Rightarrow q$
7. $p \Rightarrow (p \vee q)$
8. $(p \vee q) \Rightarrow (q \vee p)$
9. $((p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow (p \vee q \Rightarrow r)$

10. $\neg p \Rightarrow (p \Rightarrow q)$

11. $((p \Rightarrow q) \wedge (p \Rightarrow \neg q)) \Rightarrow \neg p$

The axioms of the **intuitionistic predicate calculus** are obtained by adding to the foregoing propositional axioms those in the following list, where $\forall$ and $\exists$ have their usual meanings. Note that $p[x/t]$ is the formula obtained on replacing every occurrence of x in p by t in accordance with standard conventions; see [1], pages 57-67

Predicate Axioms:

1. $\forall x(p \Rightarrow q) \Rightarrow (\forall xp \Rightarrow \forall xq)$
2. $\forall x(p \Rightarrow q) \Rightarrow (\exists xp \Rightarrow \exists xq)$
3. $p \Rightarrow \forall xp$ if x is not free in p
4. $\exists xp \Rightarrow p$ if x is not free in p
5. $\forall xp \Rightarrow p[x/t]$ if t is free for x in p
6. $p[x/t] \Rightarrow \exists xp$ if t is free for x in p
7. All generalisation of 1-6

References

[1] John L. Bell and Moshe Machover. *A Course in Mathematical Logic*. North–Holland, Amsterdam, 1977.

[2] Erret Bishop. *Foundations of Constructive Analysis*. McGraw–Hill, New York, 1967.

[3] Erret Bishop. Aspects of constructivism. In *Notes on the lectures delivered at the Tenth Holiday Mathematics Symposium*, Las Cruces, 1972. New Mexico State Univeristy.

[4] Erret Bishop. Schizophrenia in contemporary mathematic. In M. Rosenblatt, editor, *Contemporary Mathematics*, volume **39**, pages 1–32. American Mathematical Society, Providence, 1985.

[5] Erret Bishop and Douglas Bridges. *Constructive Analysis*. Grundlehren der Mathematicshen Wissenschaften **279**. Springer–Verlag, Berlin, 1985.

[6] Douglas Bridges. Constructive mathematics: A foundation for computable analysis. *Theoretical Computer Science*, 219(1):95–109, 1999.

[7] Douglas Bridges. Constructive methods in Banach algebra theory. *Mathematicae Japonicae*, 52(1):145–161, 2000.

[8] Douglas Bridges. Prime and maximal ideals in constructive ring theory. *Communication in Algebra*, 29(7):2787–2803, 2001.

[9] Douglas Bridges and Luminiţa Dediu. Paradise lost or paradise ragained? *Bulletin of the European Association for Theoretical Computer Science*, 63:141–155, 1997.

[10] Douglas Bridges and Robin Havea. A constructive analysis of a proof that the numerical range is convex. *London Mathematical Society Journal of Computation and Mathematics*, 3:191–206, 2001.

[11] Douglas Bridges and Robin Havea. A constructive version of the spectral mapping theorem. *Mathematical Logic Quaterly*, 47:299–304, 2001.

[12] Douglas Bridges and Robin Havea. Approximation to the numerical range of an element of a Banach algebra. In Laura Crosilla and Peter Schuster, editors, *From Sets and Types to Topology and Analysis: Towards a Practicable Foundation for Constructive Mathematics*, Oxford University Logic Guides, Oxford, 2005. Oxford Univeristy.

[13] Douglas Bridges and Robin Havea. Powers of a Hermitian elements in a Banach algebra. *New Zealand Journal of Mathematics*, 36:1–10, 2007.

[14] Douglas Bridges and Robin Havea. Constructive notions of maximality for ideals. *Journal of Universal Computer Science*, 14(22):3648–3657, 2009.

[15] Douglas Bridges, Robin Havea, and Peter Schuster. Ideals in constructive Banach algebra theory. *Journal of Complexity*, 22:729–737, 2006.

[16] Douglas Bridges and Fred Richman. *Varieties of Constructive Mathematics*. London Mathematical Society Lecture Notes **97**. Cambridges University, Cambridges, 1987.

[17] Douglas S. Bridges and Luminiţa Simona Vîţa. *Techniques of Constructive Analysis*. Universitext. Springer, New York, 2006.

[18] Luitzen Egbertus Jan Brouwer. *Over de Grondslagen der Wiskunde*. PhD thesis, University of Amsterdam, 1907.

[19] Michael Dummet. *Elements of Intuitionism*. Oxford Logic Guides. Oxford University Press, Oxford, 1977.

[20] Nicolas Goodman and John Myhill. Choice implies excluded middle. *Zeit. Math. Logik und Grundlagen der Math.*, 24:461, 1978.

[21] Robin Havea. *Constructive spectral and numerical range theory*. PhD thesis, University of Canterbury, Christchurch, 2001.

[22] Robin Havea. Mathematics without the law of the excluded middle. In Ian Campbell and Eve Coxon, editors, *Polynesian Paradox*. USP Institute of Pacific Studies, Suva, 2005.

[23] Robin Havea. On firmness of the state space and positive elements of a Banach algebra. *Journal of Universal Computer Science*, 11(12):1963–1969, 2005.

[24] Arend Heyting. Die formalen regeln der intuitionistischen logik. *Sitzungsber. Deutsch. Akad. Wiss. Phys.-Math. Kl.*, 16(1):42–56, 1930.

[25] Ray Mines, Fred Richman, and Win Ruitenburg. *A Course in Constructive Algebra*. Universitext. Springer–Verlag, Berlin, 1988.

[26] Fred Richman. Intuitionism as a generalization. *Philosophia Mathematica*, 5:124–128, 1990.

[27] Fred Richman. Polynomials and linear transformation. *Linear Algebra and Applications*, 131:131–137, 1990.

[28] Hiroki Takamura. An introduction to the theory of c^*–algebra in constructive mathematics. In Laura Crosilla and Peter Schuster, editors, *From Sets and Types to Topology and Analysis: Towards a Practicable Foundation for Constructive Mathematics*, Oxford University Logic Guides, Oxford, 2005. Oxford Univeristy Press.

[29] Dirk van Dalen. *Brouwer's Cambridge Lectures on Intuitionism.* Cambridges University Press, Cambridge, 1981.

[30] Dirk van Dalen. *Mystic, Geometer, and Intuitionist: The Life of L.E.J. Brouwer*, volume 1. Clarendon Press, Oxford, 1999.