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A Constructive Look at Separation Properties in Neighbourhood, Apartness, and Quasi–Apartness Spaces

A thesis presented to
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for the degree of

Master of Science

by

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August 2013

Declarations

Statement by Author

I certify that this thesis is my own work except those sections and results which have been explicitly acknowledged. I also certify that this thesis has not been previously submitted for a degree at any other institution or university.

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Abstract

In the spirit of Bishop–style constructive mathematics, the notion of an apartness space is defined before the concept of quasi–apartness space is introduced in a very natural way. This is an approach which is parallel to, or perhaps an extension of, the concept of a neighbourhood space which, introduced by Bishop, paved the way to constructivising topology at least in the general setting with points and sets as in the classical spirit. Furthermore, it is shown that there is an adjunction between the category of quasi–apartness spaces and the category of neighbourhood spaces giving a clear indication that quasi–apartness is a more natural concept when compared to apartness.

We investigate separation properties for neighbourhood spaces in some detail within a framework of constructive mathematics, and define corresponding separation properties for quasi–apartness spaces. We also deal with separation properties for spaces with inequality.

Dedicated to my Mother, Mele Pi‘ei Halaholo, who fulfilled my Father’s dreams when he passed away when I was still five years old.

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Contents

Chapter 1

Introduction

In this chapter, a brief introduction to classical and constructive mathematics are laid out for the benefit of the reader. It must be pointed out that these two areas are very broad in their own individual way but directly relevant and significant to this short research project, the reader is quickly introduced to the main concepts and underlying governing principles.

1.1 Classical Mathematics

Prior to the days of Pythagoras, mathematics was mainly used as a tool in various human affairs particularly for measuring quantities, for instance in administration records, navigation, and constructions. As Helu pointed out [22], mathematics as a science grew out to become an institution whose primary and only aim is the gaining of new knowledge leading to studying of more abstract mathematical objects.

In science, or perhaps philosophy to be more appropriate, we have literally endless accounts of debates about the notions of *existence*. What does it mean to 'exist'? Whatever the case may be, mathematics in its philosophical foundation has its fair share of attempting to answer the same question. To be more specific, a considerable number of mathematicians throughout history had attempted to define the notion of ‘mathematical existence’ which even brought out more debates amongst mathematicians [15, 17, 24, 30, 35–37].

Before the emergence of ‘modern’ constructive mathematics in the mid 20th century A.D., the practice of mathematics in general was largely dominated by classical (or traditional) (CLASS) approaches [36, 37]. Interestingly enough, the majority of mathematicians in these days and ages can rightfully be labeled as classical mathematicians. But what is a classical mathematician? The defining answer has to do with the underlying logic being used which is, in this case, classical logic; this is to be discussed further later in the thesis. However, there had been other philosophical views of mathematics, for instance Platonism and Formalism amongst others. Initiated by Plato, the former upholds that there is a world of abstract mathematical objects whose existence is independent of what we perceive in the real world and indirectly this is how they define mathematical existence. On the other hand, Hilbert and his followers believed that mathematics is a ‘formal’ game where rules are given in logic and the mathematicians must play according to the rules and the beauty of such system it is always free of contradictions. An interesting point to stress out here regarding Hilbert’s program is that logic precedes mathematics [16].

The most notable historical event in foundational debates was that between Hilbert and Brouwer at the early part of the 20th century. Since the publication of his PhD dissertation [14] in 1907, Brouwer completely objected to Hilbert’s treatment of mathematics as being a formal game and rather came up with a completely different approach which came to be known as *Brouwer’s Intuitionism.* It was a philosophy that he equally applied to mathematics with the belief that it would ultimately avoid all nonconstructivity in mathematics. It must be pointed out that Brouwer himself was a very successful classical mathematician but due to his dissatisfactions he challenged tradition from a completely different angle [37, 38]. In simple terms, Brouwer believed that mathematics is nothing else but simply a free creation of the human mind [37, 39]. The construction of the mathematical objects are freely done in the mind and from there one then abstracts the logic. Notice here that mathematics precedes logic in contrary to Hilbert's approach. The debate made its mark in the history of mathematics which climaxed when Gödel came out with his Incompleteness Theorem putting an end to Hilbert's programme [37, 39].

1.2 Constructive Mathematics

Constructive mathematics and its first appearance as an issue of interest are often associated with Kronecker [39]. As mentioned earlier, the work of Brouwer [14] brought constructivism but it still lacked formality. Heyting, Brouwer's most famous student, later wrote the axioms of intuitionistic logic[1] that appeared in [23]. Bridges and Richman in [10] discuss the three main varieties of constructive mathematics (CM) which are Brouwer's intuitionism (INT), Markov's constructivism (RUSS), and Bishop's constructivism (BISH). All these three have differences but share the common core when it comes to defining *mathematical existence*:

a mathematical object exists if and only if it is constructible,

that is there is a routine computational step–by–step procedure showing how to find the object in question. One of the key attributes always associated with CM is the complete rejection of the Law of Excluded Middle (LEM) which states that for any given statement P,

P is either true or false.

Apart from not being able to establish LEM using intuitionistic logic, it is considered the cornerstone of nonconstructivity in mathematics. In fact, this is a very powerful law in classical logic and it is heavily used when proving existence in mathematics.

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[1] See Appendix A.

To be specific, in light of LEM, an object either exists or non–exists, and to prove existence it suffices to show that assuming non–existence would lead to a contradiction. This approach shows no way of how to find the object in question. In fact, what just established is the fact that the assumption that the object's non–existence is contradictory but it does not allow us to conclude that it, therefore, exists. In short, we adopt the Richman view [31] that

CM is doing mathematics using intuitionistic logic

and

existence is equivalent to computability,

that is, if one claims the existence of an object one must be able to compute such an object. We now turn our attention to highlighting some key features of INT, RUSS, and BISH including those that are directly relevant and significant to the discussions to follow later in the thesis.

1.2.1 Intuitionistic Mathematics

In INT, the notion of an algorithm (or finite routine) is taken as primitive. Brouwer's main ideas, as summarised by Troelstra in [36, 37], are as follow.

- Mathematics is not formal; the objects of mathematics are mental constructions in the mind of the (ideal) mathematician. Only the thought constructions of the (idealised) mathematician are exact.

- Mathematics is independent of experience in the outside world, and mathematics is in principle also independent of language. Communication by language may serve to suggest similar thought constructions to others, but there is no guarantee that these constructions are the same.

- Mathematics does not depend on logic; on the contrary, logic is part of mathematics.

Brouwer is well known with his famous two principles which he heavily utilised in many of his proofs. Firstly, the usage of:

Choice Sequence: These are sequences of natural numbers which may be either lawlike or lawless.

Lawlike has to do with some predetermined rule whereas being lawless has to do with 'arbitrarily creative subject'. The intuitionistic interpretation together with choice sequences led Brouwer to formulate his *continuity axioms*, which asserts that a natural number valued–function f defined on a collection of choice sequences depends on a finite initial segment of the sequence.

Note that this intuitionistic principle is rarely welcomed in classical mathematics.

The Fan Theorem: *Every detachable bar of a fan in uniform.*

The Fan theorem is equivalent to the statement that the Cantor space $2^{\mathbb{N}}$ is compact[2]. Further, the Fan theorem is well matched with classical logic but it is not valid in Constructive Recursive Mathematics [37, 38].

1.2.2 Constructive Recursive Mathematics

As in INT, RUSS takes the notion of algorithm as primitive but with a slight twist. Their algorithm is a more concrete form of constructivism in which the objects in mathematics are sequence of words in different alphabets. A proof in RUSS is an algorithm that operates on the fixed programming language. For instance, to prove 'A implies B', one must give an algorithm which given a proof of A, it can be turned into a proof of B.

RUSS is well known with its adoption of

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[2] We take here the Cantor space to be countably infinite topological product of the discrete 2–point space $\{0, 1\}$, that is 2 denotes the 2–element set $\{0, 1\}$ with the discrete topology.

Markov's Principle (MP) If (a_n) is a binary sequence[3] for which it is contradictory that all terms be 0, then there exists n such that $a_n = 1$.

a statement that is not freely acceptable in all the other varieties apart from CLASS. It is considered a form of unbounded search meaning that if a construction does not run forever then it must terminate after some finite number of steps.

Lastly, it is worth pointing out at this stage that

Church's Thesis (CT) If every total function $f : \mathbb{N} \to \mathbb{N}$ is recursive, then there exists $m \in \mathbb{N}$ such that the m^{th} recursive function f_m is total and $f_m(n) = f(n)$ for each n.

is not acceptable in CLASS and so as in INT since it is not compatible with the Fan Theorem [4].

1.2.3 Bishop's Constructive Mathematics

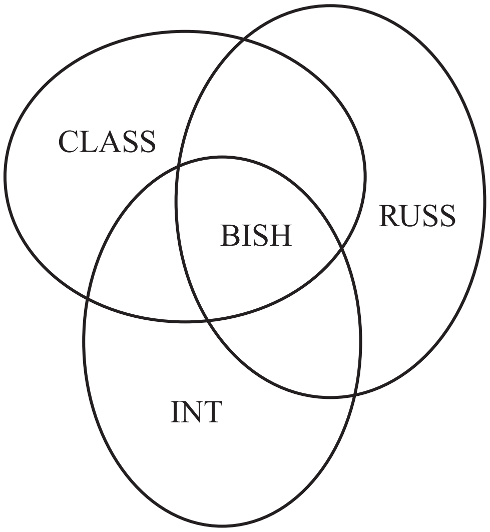
In BISH, the notion of algorithm or finite routine is taken as primitive. The existence of mathematical objects is equivalent to computability. Bishop's programme is guided by the following three principles as stated by Troelstra in [36].

1. Avoid concepts defined in a negative way.
2. Avoid defining irrelevant concepts, that is, among the many possible classically equivalent, but constructively distinct definitions of a concept, choose the one or two which are mathematically fruitful ones, and disregard the others.
3. Avoid pseudo–generality, that is, do not hesitate to introduce an extra assumption if it facilitates the theory and the examples one is interested in satisfying the assumption.

---

[3] A ***binary sequence*** is a finite routine that assigns to each positive integer n an element of $\{0, 1\}$. Let a be a binary sequence.

One of the key advantages of choosing BISH over INT and RUSS is the fact that every theorem in BISH is automatically a theorem in INT and RUSS let alone CLASS [10]. In other words, BISH is consistent with CLASS that every proposition P in BISH has an immediate interpretation in CLASS, and a proof of P in BISH is also a proof of P in CLASS. This is not valid in INT and RUSS. Further, every proof of a proposition P in BISH is a proof of P in INT and RUSS. On the whole, BISH can be seen as a common core of INT, RUSS, and CLASS, and that INT and RUSS can be regarded as extensions of BISH.



For a working mathematician in CLASS, it is very easy to read BISH without much distortion due to technicalities. For example, opening Bishop's monograph [5] or [6], one immediately realises that it is about analysis just like as in a typical textbook in CLASS. This is not often the case with INT and RUSS where one has to be very familiar with the technicalities and terminologies in order to appreciate the presentation. As such, ***throughout this thesis, the presentation will be carried out completely within the framework of* BISH**.

Bishop did not give a clear detailed account of a formal system on which his version of constructivism should be based. There are two possible foundations account for BISH that emerged in the first half of the 1970s.

1. Martin–Löf's Intuitionistic Type Theory (ITT).[4]

2. Myhill's Constructive Set Theory (CST).

Aczel's Constructive Zermelo–Fraenkel Set Theory CZF can be seen as providing a foundation for BISH but can also be regarded as an extension of CST [3].

1.3 Interpretations and Principles

In this section we look at formal intuitionistic interpretations of the logical connectives, quantifications, and negations. We will also look at well known principles considered to be highly nonconstructive.

The following are the *BHK-interpretations* of the logical connectives and quantifiers. Let P and Q be any given statements. These constructive interpretations enabled Heyting to produce a complete list of the axioms of intuitionistic logic [23].

$\wedge$ **(and)**: To prove $\mathsf{P} \wedge \mathsf{Q}$ (P and Q), we must have a proof of P and a proof of Q.

$\vee$ **(or)**: To prove $\mathsf{P} \vee \mathsf{Q}$ (P or Q), we must have either a proof of P or a proof of Q.

$\Rightarrow$ (**implies**): To prove $\mathsf{P} \Rightarrow \mathsf{Q}$ (P implies Q) means there is an algorithm that transforms a proof of P into a proof of Q.

$\neg$ **(not)**: We interpret $\neg\mathsf{P}$ (Not P) as $\mathsf{P} \Rightarrow \mathsf{Q}$, where Q is a contradiction such as $0 = 1$.

---

[4] In [27], ITT is presented and also gave a foundation for a number of development including formal topology. It is a dependent type theory which includes basis types such as the natural numbers, plus with dependent types such as type-indexed products $(\prod x : A)B(x)$ and sums $(\sum x : A)B(x)$.

$\exists$ **(exists)**: To prove $\exists \mathsf{a} \in \mathsf{A}\mathsf{P}(\mathsf{a})$ (There exists a such that a has the property P), we must compute a and demonstrate that $\mathsf{P}(\mathsf{a})$ holds.

$\forall$ **(for all)**: A proof of $\forall \mathsf{a} \in \mathsf{A}\mathsf{P}(\mathsf{a})$ (For all a in A, a has the property P) is an algorithm that, applied to each element a of A and to the data showing that a belongs to A, proves that $\mathsf{P}(\mathsf{a})$ holds.

The interpretation of $\mathsf{P} \wedge \mathsf{Q}$ is similar to the classical treatment. Classically, to prove $\mathsf{P} \vee \mathsf{Q}$ it suffices to establish $\neg(\neg\mathsf{P} \wedge \neg\mathsf{Q})$; but proving the latter is not enough to prove the former in constructive mathematics. Why? Generally, in constructive mathematics it is not possible to decide, from a proof of $\neg(\neg\mathsf{P} \wedge \neg\mathsf{Q})$, which of the alternatives P, Q holds. The constructive interpretation of $\vee$ is well tied to the notion of decidability in constructive mathematics; one of the main features of constructive mathematics is being able to make decision and the constructive interpretation of $\vee$ captures it all [10].

To prove $\exists \mathsf{a} \in \mathsf{P}(\mathsf{a})$ in the classical way, it suffices to show that $\neg\forall \mathsf{a} \neg \mathsf{P}(\mathsf{a})$; classical existence is equivalent to the impossibility of nonexistence. In constructive mathematics to prove $\exists x \in \mathsf{P}(x)$ we must construct an object ξ (at least in principle), show that ξ satisfies the conditions for membership of A, and then show that $\mathsf{P}(\xi)$ holds [10].

The essence of our view of 'constructive proof' is no different from how a computer programmer views a series of commands written in some specified language simply as a set of instructions. Given such a set of instructions a machine can follow the instructions on how to construct or find whatever object under consideration. Thus, constructive mathematics can affectionately be seen as honest mathematics simply because if one claims the existence of an object then he or she should provide instructions on how to find such an object.

Interpreting existence in this way means we have to question the constructive status of many well–known classical results. For instance, the LEM as mentioned above which cannot be proved constructively. Thus LEM is unacceptable in constructive mathematics although it is one of the main roots of nonconstructively in both mathematics and not to mention computer science [20]. To make this clearer, let us look at the following.

Let a be a binary sequence and consider the following statements.

$$\begin{aligned} \mathsf{P}(a): &\quad a_n = 1 \text{ for some } n \\ \neg\mathsf{P}(a): &\quad a_n = 0 \text{ for all } n \\ \mathsf{P}(a) \vee \neg\mathsf{P}(a): &\quad \text{Either } \mathsf{P}(a) \text{ or } \neg\mathsf{P}(a) \\ \forall a(\mathsf{P}(a) \vee \neg\mathsf{P}(a)): &\quad \text{For all } a\text{, either } \mathsf{P}(a) \text{ or } \neg\mathsf{P}(a) \end{aligned}$$

Classically, to prove the disjunction $\mathsf{P}(a) \vee \neg\mathsf{P}(a)$, it suffices to show that $\mathsf{P}(a)$ and $\neg\mathsf{P}(a)$ cannot be both false. On the contrary, a constructive proof of $\mathsf{P}(a) \vee \neg\mathsf{P}(a)$ involves a finite routine or algorithm which either show that $a_n = 0$ for all n, or construct a positive integer n with $a_n = 1$. The constructive meaning of disjunction which entails from the constructive interpretation of existence such that $\mathsf{P}_1 \vee \mathsf{P}_2$ holds if and only if there exists P_i. To prove $\mathsf{P}_1 \vee \mathsf{P}_2$, it is not enough to show that P_1 and P_2 cannot both be false.

Now, the strict interpretation of existence in constructive mathematics, as a result, leads to the rejection of proof by contradiction of $\neg\neg\mathsf{P} \Rightarrow \mathsf{P}$ simply because of the unacceptability of LEM from which it is derived. Although LEM cannot be established constructively, it was believed by most classical mathematicians at the time of 'the foundational debate' that it was impossible to carry out any serious mathematics without LEM as the formalist purported [9]:

> *No-one, though he speaks with the tongues of angels, will keep people from using the law of excluded middle.*

In CM, it is very common to produce Brouwerian (counter)examples to demonstrate how far one can expect a result to remain constructive. We say that a is a Brouwerian counterexample to the statement $\forall a(P(a) \vee \neg P(a))$ meaning that it is not a counterexample in the usual sense but it is an evidence that a statement does not admit a constructive proof in the context of constructive mathematics [10]. The following statements are considered highly nonconstructive and very handy when constructing Brouwerian (counter)examples. They were extensively used by Brouwer but we choose to adopt the names used by Bishop [5, 10].

Limited Principle of Omniscience(LPO): For any binary sequence $(a_n)_{n\geq 1}$, either $a_n = 0$ for all n, or there exists n such that $a_n = 1$.

Weak LPO (WLPO): For any binary sequence $(a_n)_{n\geq 1}$, either $a_n = 0$ for all n, or not $a_n = 0$ for all n.

Lesser LPO (LLPO): For any binary sequence $(a_n)_{n\geq 1}$ with at most one non–zero term, either $a_n = 0$ for all even n or $a_n = 0$ for all odd n.

It follows that for any classical proposition that implies either any of these omniscience principles, it is regarded as essentially nonconstructive. Let us look at some Brouwerian examples to illustrate the discussions. The first example is due to Myhill [28] which shows the dubious nature of the full–blooded **Axiom of Choice (AC)**: *Let* A *and* B *be sets, and* C *be a subset of* $A \times B$ *such that for each* $a \in A$ *there exists* $b \in B$ *such that* $(a, b) \in C$*. Then there exists a 'choice function'* $f : A \to B$ *such that* $(a, f(a)) \in C$ *for all* $a \in A$*.*

Proposition 1.3.1 AC *implies* LEM.

Proof. Let P be any constructively meaningful statement and define the set $A = \{s, t\}$ together with the equality relation given by $s = t \Leftrightarrow P$ holds. Consider now the set $B = \{0, 1\}$ with the standard equality, and let $S = \{(s, 0), (t, 1)\} \subset A \times B$, with the equality relation derived from those on A and B:

$$(x, y) =_{A\times B} (x_1, y_1) \quad \Leftrightarrow \quad x =_A x_1 \in A \wedge y =_B y_1 \in B.$$

Assume that there exists a function $f : A \to B$ such that $(x, f(x)) \in S$ for all $x \in A$. If $f(s) = 1$ or $f(t) = 0$, then $s = t$ and hence P holds; if $f(s) = 0$ and $f(t) = 1$, then $\neg(s = t)$ and therefore $\neg P$ holds. Thus we have derived $P \vee \neg P$ from AC. ■

The next three examples are presented following the style Bridges presents in [8]. The first one shows that the classical statement 'For any nonnegative real number, it is either zero or positive' is nonconstructive.

Proposition 1.3.2 $\forall x \in \mathbb{R}(x \geq 0 \Rightarrow x > 0 \vee x = 0)$ *implies* LPO

Proof. Suppose that $\forall x \in \mathbb{R}(x \geq 0 \Rightarrow x > 0 \vee x = 0)$ holds, and consider any binary sequence (a_n). Define

$$x_n = \begin{cases} 1/k & \text{if } k \leq n, a_j = 0 \text{ for all } j < k, \text{ and } a_k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_n)$ is a nonnegative real number. If $x > 0$, then there exists N such that $x_N > 1/N$; by check terms $a_1, \ldots, a_N$ we can then find $k < N$ such that $a_k = 1$. If $x = 0$, then $a_n = 0$ for all n. Thus we have derived LPO from $\forall x \in \mathbb{R}(x \geq 0 \Rightarrow x > 0 \vee x = 0)$. ■

Proposition 1.3.3 $\forall x \in \mathbb{R}(x \geq 0 \vee x \leq 0)$ *implies* LLPO.

Proof. Let $(a_n)_{n \geq 1}$ be a binary sequence with at most one term equal to 1, and define a real number by

$$x = \left\{ \left(-\frac{1}{n}, \frac{1}{n} \right) : \forall k \leq n \, (a_k = 0) \right\} \cup \left\{ \left((-1)^n \frac{1}{n}, (-1)^n \frac{1}{n} \right) : a_n = 1 \right\}.$$

If $x \geq 0$, then it is impossible that $a_n = 1$ for an odd n, so $a_n = 0$ for all odd n. Likewise, if $x \leq 0$, then $a_n = 0$ for all even n. Thus we have derived LLPO from $\forall x \in \mathbb{R}(x \geq 0 \vee x \leq 0)$. ■

Proposition 1.3.4 $\forall x, y \in \mathbb{R}(\neg(x \geq y) \Rightarrow y > x)$ *implies* MP

Proof. Assume that $\forall x, y \in \mathbb{R}(\neg(x \geq y) \Rightarrow y > x)$ holds. Let $(a_n)_{n\geq 1}$ be an increasing binary sequence such that $\neg\forall n\,(a_n = 0)$, and define a real number by

$$x = \left\{\left(0, \frac{1}{n}\right) : a_n = 0\right\} \cup \left\{\left(\frac{1}{n}, \frac{1}{n}\right) : a_n = 1 - a_{n-1}\right\}.$$

Then $\neg(0 \geq x)$: for if $q \leq 0$ for all $(q, q') \in x$, then $a_n = 0$ for all n, a contradiction. It follows from $\forall x, y \in \mathbb{R}(\neg(x \geq y) \Rightarrow y > x)$ that $x > 0$ and hence that there exists $(q, q') \in x$ such that $q > 0$. Then $(q, q') = \left(\frac{1}{n}, \frac{1}{n}\right)$ for a(unique) n such that $a_n = 1 - a_{n-1}$. Thus we derived MP from $\forall x, y \in \mathbb{R}(\neg(x \geq y) \Rightarrow y > x)$. ■

1.4 Constructive Topology

We devote this section to giving a brief introduction to how topology had been developed particularly in the framework of Bishop's constructive mathematics. This, however, will naturally lead to further discussions to follow in the next two chapters of the thesis.

In [6], Bishop dismissed topology with the following remark:

> Very little is left of general topology after that vehicle of classical mathematics has been taken apart and reassembled constructively. With some regret, plus a large measure of relief, we see this flamboyant engine collapse to constructive size. (See page 63 of [4])

Clearly, Bishop himself knew the challenging nature of constructivising general topology and this area of constructive mathematic had been lying dormant for a number of years. However, in recent years the works of Bridges and Vîţa [11, 12] on axiomatic constructive theory on nearness spaces based on primitive notions of point–set nearness and apartness had sparked interest let alone proved very promising. There are other constructive treatments of topology which can be seen in the works of Grayson [18, 19] on general topology, Sambin [33, 34] for formal topology in Martin-Löf type theory [27], and Aczel [2] for topology in CZF. Additionally, Palmgren and Schuster in [29] investigate interesting relationships between apartness and formal topology.

Throughout, we shall adopt Bishop's conventions as they appear in [5]. This gives a very natural setting for the development that came later after a considerable number of years. Furthermore, we shall assume familiarity with classical topology as presented in [18, 19, 26].

We define a ***neighbourhood space*** as a pair (X, τ) consisting of a set X and a set τ of subsets of X such that

NS1 $\forall x \in X \exists U \in \tau (x \in U)$,

NS2 $\forall x \in X \forall U, V \in \tau [x \in U \cap V \Longrightarrow \exists W \in \tau (x \in W \subset U \cap V)]$.

The set τ is called an ***open base*** on X. The ***interior***, S°, of a subset S of X is the set

$$S^\circ := \{x \in S | \exists U \in \tau (x \in U \subset S)\}.$$

A subset S of X is ***open*** if $S = S^\circ$. Every metric space (X, d) induces a neighbourhood structure defined by

$$\tau := \{B(x, r) | x \in X \wedge r > 0\},$$

where $B(x, r)$ is the ***open ball***

$$B(x, r) := \{y \in X | d(x, y) < r\}$$

centered at x with radius r.

About thirty years later, Bridges and Vîţa [11, 12] proposed a program of *apartness spaces* as an alternative approach to topology from a constructive point of view. The requirement is that the set X under consideration must come furnished with an inequality $\neq$ relation such that

- $x \neq y \Rightarrow y \neq x$, and

- $x \neq y \Rightarrow \neg(x = y)$.

We say that $\neq$ is ***nontrivial*** if there exists $a, b \in X$ with $a \neq b$. For a subset S of X, the ***complement of*** S, denoted by $\sim S$, is the set

$$\sim S := \{x \in X | \forall y \in S (x \neq y)\}.$$

A (point–set) ***apartness space*** is a pair $\langle X, - \rangle$ consisting of the set X with a nontrivial inequality $\neq$, and with an operation, $-$, on the subsets of X such that for all $x, y \in X$ and for all sets $S, T \subset X$, the following axioms hold.

A1 $x \neq y \Rightarrow x \in -\{y\}$

A2 $-S \subset \sim S$

A3 $-(S \cup T) = -S \cap -T$

A4 $-S \subset \sim T \Rightarrow -S \subset -T$

A5 $x \in -S \Rightarrow \forall y \in X (x \neq y \vee y \in -S)$

We say that $-$ is an ***apartness on*** X. Every nontrivial metric space (X, d) has a nontrivial inequality $\neq$ and is defined by $x \neq y := d(x, y) > 0$. Furthermore, the metric space X also induces an ***apartness structure*** defined by

$$-S := \{x \in X | \exists r > 0 \forall y \in S (r \leq d(x, y))\}.$$

For a subset S of a set X, the ***logical complement***, denoted by $\neg S$, is the set

$$\neg S := \{x \in X | \neg(x \in S)\}.$$

For each neighbourhood space (X, τ), one can define an operation $-_\tau$ on the subsets of X by taking $-_\tau S$ to be $(\neg S)^\circ$. We now look at two examples showing that, in general, this operation is not an apartness on X as it need not satisfy either $A1$ nor $A5$ as appeared in [25].

Example 1.4.1 $-_\tau$ need not satisfy $A1$. To see this, let $\Sigma := \{0, 1\}$ with $0 \neq 1$ be the Sierpinski space whose open base is $\sigma := \{\{0\}, \Sigma\}$. (This neighbourhood space is classically T_0 but not T_1). The operation $-_\tau$ induced by σ does not satisfy $A1$. Indeed, since $0 \neq 1$, condition $A1$ would imply $1 \in -_\sigma\{0\} = \emptyset$, which is impossible. (Actually $A1$ forces spaces to be T_1; see [21].) ☺

Example 1.4.2 $-_\tau$ need not satisfy $A1$. To see this, let σ be the discrete neighbourhood space on the set $\mathbb{R}$ of real numbers. (The discrete neighbourhood space is classically T_0 but not T_1). Clearly, the operation $-_\tau$ induced by σ is nothing but the logical complement: that is, $-_\delta S = \neg S$ for all $S \subset \mathbb{R}$. If $A5$ holds for this operation, and with $x \neq y := \neg(x = y)$, then we can derive WLPO in the form

$$\forall x \in \mathbb{R}(x = 0 \vee \neg(x = 0)).$$

To see this, set $S := \neg\{0\}$, for which $0 \in -S - \neg\{0\} = \neg\neg\{0\}$. For each $x \in \mathbf{R}$, by $A5$, we either have $x \neq 0$ or $x \in -S$: that is, either $\neg(x = 0)$, or $\neg\neg(x = 0)$ which is equivalent to $x = 0$. Since it is doubtful that we can achieve a constructive proof of WLPO, we cannot expect to find one of $A5$ for $-_\delta$. ☺

Because of these examples, we introduce the notion of a quasi–apartness space by dropping conditions $A1$ and $A5$ from the definition of an apartness space. Furthermore, we also drop off the inequality. Following the work of Ishihara *et al.* in [25], it can be shown that each neighbourhood space induces a quasi–apartness from which we can construct the weakest and also the strongest neighbourhood. Richman in [32] has shown that requiring inequality on an apartness space is superfluous. Moreover, neighbourhood spaces do not come with an inequality. Therefore we do not include inequality in the definition of a quasi–apartness space.

A ***quasi–apartness space*** is a pair $\langle X, - \rangle$ consisting of a set X with an operation $-$ on the subsets of X such that for all sets $S, T \subset X$ the following hold.

QA1. $-\emptyset = X$

QA2. $-\mathsf{S} \subset \neg\mathsf{S}$

QA3. $-(\mathsf{S} \cup \mathsf{T}) = -\mathsf{S} \cap -\mathsf{T}$

QA4. $-\mathsf{S} \subset \neg\mathsf{T} \Rightarrow -\mathsf{S} \subset -\mathsf{T}$

We say that $-$ is a ***quasi–apartness*** on X. The notion of a quasi–apartness is more general than the notion of a (point–set) apartness in the sense of Bridges and Vîţă [11–13].

Chapter 2

Quasi–Apartness and Neighbourhood Spaces

2.1 Introduction

In this chapter, we introduce the notion of quasi–apartness and neighbourhood spaces. As mentioned earlier, Bishop in [5, 6] pointed out the challenging aspects of constructivising topology and introduced the notion of neighbourhood spaces. The presentation throughout this chapter follows along the same spirit particularly the works of Ishihara *et al.* in [25] and classic works of Grayson in [18, 19].

2.2 From Neighbourhood Spaces to Quasi–apartness Spaces

We first note that in the definition of a quasi–apartness presented in the preceding chapter, if $\mathsf{S} \subset \mathsf{T}$ then $-\mathsf{T} \subset -\mathsf{S}$, by QA3. Since $\mathsf{S} \subset \mathsf{T} \to \mathsf{T} = \mathsf{S} \cup \mathsf{T}$, we have $-\mathsf{T} = -(\mathsf{S} \cup \mathsf{T}) = -\mathsf{S} \cap -\mathsf{T} \subset -\mathsf{S}$. Therefore $\mathsf{S} \subset \mathsf{T} \to -\mathsf{T} \subset -\mathsf{S}$. As it appears in [25], axioms QA2, QA3 and QA4 correspond to the axioms C2, C3, and C4 in [32], respectively.

Ishihara *et al.* in [25] deal with constructing quasi–apartness structures on neighbourhood spaces. Following their work, there would not be any reference to the powerset axiom and the construction of a quasi-apartness structure on a neighbourhood space shall be in a predicative way.

The following proposition is a straightforward adaptation of the results in [32] which asserts that an apartness space is quasi–apartness but, in general, the converse does not hold. We'll deal with a partial answer to this later in section 2.3.

Proposition 2.2.1 *Let* $\langle X, -\rangle$ *be an apartness space. Then it is a quasi-apartness space.*

Proof. We verify all the axioms.

QA1: We first dispose of QA1. Since by definition of apartness spaces, the inequality on X is nontrivial, $\exists a, b \in X$ such that $a \neq b$, and hence $a \in -\{b\}$ (or $b \in -\{a\}$) by A1. Let $x \in X$. Then $a \neq b$ or $x \in -\{b\}$ by A5, and hence $x \in -\{a\}$ or $x \in -\{b\}$ by A1. For each $z \in X$, since $\emptyset \subset \{z\} \rightarrow -\{z\} \subset -\emptyset$ by A3, and $-\{b\}, -\{a\} \in -\emptyset$. Therefore $x \in -\emptyset = X$.

QA2: First notice that $\sim S \subset \neg S$. Suppose that $-S \subset \sim S$ by A2 and let $x \in -S$. Then $x \in \sim S$ and hence $x \in \neg S$. Thus we have $-S \subset \neg S$.

QA3: This is trivial since A3 is equivalent to QA3.

QA4: Suppose that $-S \subset \neg S$, and let $x \in -S$. Then for each $y \in T$, either $x \neq y$ or $y \in -S$ by A5. Since $y \in -S \subset T$ is ruled out; impossible because $-S \subset \neg T$ and $-S \subset T$ are contradictory, that is $T \vee \neg T$ implies LEM. The latter case, we have $y \in -S \subset \neg T$. Thus $x \in \sim T$. Therefore, since $-S \subset \sim T$, we have $-S \subset -T$ by A4, and so $x \in -T$. ■

Proposition 2.2.2 *Let* (X, τ) *be a neighbourhood space, and define an operation* $-_\tau$ *on the subsets of* X *by letting* $-_\tau S := (\neg S)^\circ$. *Then* $\langle X, -_\tau \rangle$ *is a quasi-apartness space.*

Proof. We verify all the axioms.

QA1: By NS1, for each $x \in X \exists U \in \tau$ such that $x \in U \subset \neg\emptyset$. Since $-_\tau\emptyset := (\neg\emptyset)^\circ \subset \neg\emptyset$, we have $x \in U \subset -_\tau\emptyset$. Hence $-_\tau\emptyset = X$.

QA2: Suppose that $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$. Since $-_\tau S = (\neg S)^\circ \subset \neg S$, we have $x \in U \subset -_\tau S \subset \neg S$. Thus $-_\tau S \subset \neg S$.

QA3: Suppose that $x \in -_\tau(S \cup T)$. Then there exists $U \in \tau$ such that $x \in U \subset \neg(S \cup T) = \neg S \cap \neg T$, and hence $x \in U \subset \neg S$ and $x \in U \subset \neg T$. Since $-_\tau S = (\neg S)^\circ \subset \neg S$ and $-_\tau T = (\neg T)^\circ \subset \neg T$, we have $x \in U \subset -_\tau S$ and $x \in U \subset -_\tau T$. Therefore $x \in U \subset -_\tau S \cap -_\tau T$. Thus $x \in -_\tau S \cap -_\tau T$. Conversely, suppose that $x \in -_\tau S \cap -_\tau T$. Then there exists $U, V \in \tau$ such that $x \in U \subset \neg S$ and $x \in V \subset \neg T$, and hence there exists $W \in \tau$ such that $x \in W \in U \cap V \subset \neg S \cap \neg T = \neg(S \cup T)$, by NS2. Since $-_\tau(S \cup T) = (\neg(S \cup T))^\circ \subset \neg(S \cup T)$, we have $x \in W \in U \cap V \subset -_\tau(S \cup T)$. Therefore $x \in -_\tau(S \cup T)$. Thus $-_\tau(S \cup T) = -_\tau S \cap -_\tau T$.

QA4: Suppose that $-_\tau S \subset \neg T$ and $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$. Since $y \in U \subset \neg S$ for all $y \in U$, we have $U \subset -_\tau S$, and hence $x \in U \subset \neg T$. Therefore $x \in -_\tau T$. ■

Lemma 2.2.3 *Let* $\langle X, - \rangle$ *be a quasi–apartness space. Then* $-\neg - S = -S$.

Proof. By QA2, we have $- - S \subset \neg - S$. Since $- - S = S$, $S \subset \neg - S$. Therefore $-\neg - S \subset -S$ by QA3. It now follows that $S \subset \neg\neg S$ and hence $-S \subset \neg\neg - S$. Therefore $-S \subset -\neg - S$ by QA4. Thus $-S = -\neg - S$. ■

The preceding lemma is very useful when proving some of the propositions to follow. Notice that classically we could assert that $S = \neg\neg S$. However, constructively $S \subset \neg\neg S$ and $S = - - S$ can hold but $S = \neg\neg S$ implies LEM [25].

Definition 2.2.4 Let $-$ and $-'$ be two quasi–apartness operators on a set X. Then we say that:

- $-$ *is weaker than* $-'$ (or $-'$ *is stronger than* $-$) and write $- \preceq -'$ if

$$\forall S \subset X(-S \subset -'S).$$

- $-$ and $-'$ are *equivalent*, and write $- \simeq -'$, if $- \preceq -'$ and $-' \preceq -$. ▲

Definition 2.2.5 Let τ and τ' be two open bases on a set X. Then we say that:

- τ is *weaker* than τ' (or τ' is *stronger* than τ), and write $\tau \sqsubseteq \tau'$, if

$$\forall x \in X \forall U \in \tau[x \in U \to \exists V \in \tau'(x \in V \subset U)].$$

- τ and τ' are *equivalent*, and write $\tau \approx \tau'$, if $\tau \sqsubseteq \tau'$ and $\tau' \sqsubseteq \tau$. Note $\tau \approx \tau'$ if and only if τ and τ' give the same open sets. ▲

In classical topology the concepts of coarser and finer topologies are captured in the above definitions where we choose to work with quasi–apartness spaces and neighbourhood spaces.

Lemma 2.2.6 *Let τ and τ' be open bases on a set X. If $\tau \sqsubseteq \tau'$, then $-_\tau \preceq -_{\tau'}$.*

Proof. Suppose that $\tau \sqsubseteq \tau'$ and let $x \in -_\tau S$. Then there exists $U \in \tau$ and $x \in U$ such that $x \in U \subset \neg S$, and hence there exists $V \in \tau'$ such that $x \in V \subset U \subset \neg S$. Since $-S = (\neg S)^\circ \subset \neg S$, we have $x \in V \subset U \subset -_\tau S \subset -_{\tau'} S$. Thus $-_\tau \preceq -_{\tau'}$. ■

Lemma 2.2.7 *Let τ and τ' be open bases on a set X. Then $-_\tau \preceq -_{\tau'}$ if and only if for each $U \in \tau$ and $x \in U$ there exists $V \in \tau'$ such that $x \in V \subset \neg\neg U$.*

Proof. Suppose that $-_\tau \preceq -'_\tau$. Then given $U \in \tau$ and $x \in U$, since $U \subset \neg\neg U$, we have $x \in U \subset -_\tau \neg U \subset -_{\tau'} \neg U$, and hence there exists $V \in \tau'$ such that $x \in V \subset \neg\neg U$.

Conversely suppose that for each $U \in \tau$ and $x \in U$ there exists $V \in \tau'$ such that $x \in V \subset \neg\neg U$, and let $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$, and hence there exists $V \in \tau'$ such that $x \in V \subset \neg\neg U$. Since $U \subset \neg S \to \neg\neg U \subset \neg S$ and $-S := (\neg S)^\circ \subset \neg S$, we have $x \in V \subset \neg\neg U \subset -_\tau S \subset -_{\tau'} S$, and so $x \in -_{\tau'} S$. Thus $-_\tau \preceq -'_\tau$. $\blacksquare$

Proposition 2.2.8 *Let* τ *and* τ' *be open bases on a set* X. *If* $-_\tau \preceq -_{\tau'}$, *then there exists an open base* σ *such that* $\tau \sqsubseteq \sigma$ *and* $-_{\tau'} \simeq -_\sigma$.

Proof. Suppose that $-_\tau \preceq -_{\tau'}$, and let $\{U \cap V \mid U \in \tau \wedge V \in \tau'\}$. Then, given $U \in \tau$ and $x \in U$ there exists $V \in \tau'$ such that $x \in V \subset \neg\neg U$, by Lemma 2.2.7, and hence there exists $W \in \sigma$ such that $x \in W \subset V \subset \neg\neg U$, by Lemma 2.2.6. Since $-\neg U = (\neg\neg U)^\circ \subset \neg\neg U$ and $U \subset \neg\neg U$, we have $x \in W \subset V \subset -_\tau \neg U \subset -_{\tau'} \neg U \subset -_\sigma \neg U$. Thus $-_\tau \preceq -_{\tau'} \preceq -_\sigma$.

To show that $-_\sigma \preceq -_{\tau'}$, let $x \in U \cap V$ for some $U \in \tau$ and $V \in \tau'$. Then by the hypothesis and Lemma 2.2.7, there exists $W \in \tau'$ such that $x \in W \subset \neg\neg U$, and hence

$$x \in W \cap V \subset \neg\neg U \cap V \subset \neg\neg U \cap \neg\neg V = \neg\neg(U \cap V).$$

Therefore there exists $W' \in \tau'$ such that $x \in W' \subset (W \cap V) \subset \neg\neg(U \cap V)$. Since $-\neg(U \cap V) = \{\neg\neg(U \cap V)\}^\circ \subset \neg\neg(U \cap V)$, we have $x \in W' \subset (W \cap V) \subset -_\sigma \neg(U \cap V) \subset -_{\tau'} \neg(U \cap V)$. Thus $-_\tau \preceq -_\sigma \preceq -_{\tau'}$. $\blacksquare$

Definition 2.2.9 A function f between quasi–apartness spaces $\langle X, -\rangle$ and $\langle Y, -'\rangle$ is *continuous* if for all $x \in X$ and $S \subset X$,

$$f(x) \in -'f(S) \to x \in -S.$$

$\blacktriangle$

Definition 2.2.10 A function f between neighbourhood spaces (X, τ) and (Y, τ') is *continuous* if for all $x \in \mathsf{X}$ and $\mathsf{V} \in \tau'$, we have

$$\mathsf{f}(x) \in \mathsf{V} \to \exists \mathsf{U} \in \tau \left(x \in \mathsf{U} \subset \mathsf{f}^{-1}(\mathsf{V})\right).$$

▲

Theorem 2.2.11 *Let* f *be a mapping between neighbourhood spaces* (X, τ) *and* (Y, τ'). *Then* $\mathsf{f} : \langle \mathsf{X}, -_\tau \rangle \to \langle \mathsf{Y}, -_{\tau'} \rangle$ *is continuous if and only if there exists an open base* σ *with* $-_\tau \simeq -_\sigma$ *such that* $\mathsf{f} : (\mathsf{X}, \sigma) \to (\mathsf{Y}, \tau')$ *is continuous.*

Proof. Suppose that $\mathsf{f} : (\mathsf{X}, \sigma) \to (\mathsf{Y}, \tau')$ is continuous for some open base σ with $-_\tau \simeq -_\sigma$, and let $\mathsf{f}(x) \in -_{\tau'}\mathsf{f}(\mathsf{S})$. Then there exists $\mathsf{V} \in \tau'$ such that $\mathsf{f}(x) \in \mathsf{V} \subset \neg\mathsf{f}(\mathsf{S})$, and hence there exists $\mathsf{U} \in \sigma$ such that $x \in \mathsf{U} \subset \mathsf{f}^{-1}(\mathsf{V})$. Therefore $x \in \mathsf{U} \subset \mathsf{f}^{-1}(\mathsf{V}) \subset \mathsf{f}^{-1}(\neg\mathsf{f}(\mathsf{S})) = \neg\mathsf{f}^{-1}(\mathsf{f}(\mathsf{S})) \subset \neg\mathsf{S}$, and so $x \in -_\sigma S = -_\tau S$. Thus $\mathsf{f} : \langle \mathsf{X}, -_\tau \rangle \to \langle \mathsf{Y}, -_{\tau'} \rangle$ is continuous.

Conversely suppose that $\mathsf{f} : \langle \mathsf{X}, -_\tau \rangle \to \langle \mathsf{Y}, -_{\tau'} \rangle$ is continuous, and let $\tau_\mathsf{f} := \left\{\mathsf{f}^{-1}(\mathsf{U}) \mid \mathsf{U} \in \tau'\right\}$. Then τ_f is an open base on X. Note that if σ is an open base on X, then $\mathsf{f} : (\mathsf{X}, \sigma) \to (\mathsf{Y}, \tau')$ is continuous if $\tau_\mathsf{f} \sqsubseteq \sigma$. To show that $-_{\tau_\mathsf{f}} \preceq -_\tau$, assume that $x \in \mathsf{f}^{-1}(\mathsf{U})$ for some $\mathsf{U} \in \tau'$. Then letting $\mathsf{T} := \mathsf{f}^{-1}(\neg\mathsf{V})$, since $\mathsf{f}(\mathsf{T}) \subset \neg\mathsf{U}$, we have $\mathsf{f}(x) \in \mathsf{U} \subset \neg\neg\mathsf{U} \subset \neg\mathsf{f}(\mathsf{T})$, and hence $\mathsf{f}(x) \in -_{\tau'}\mathsf{f}(\mathsf{T})$. Therefore, since $x \in \mathsf{f}^{-1}(\mathsf{U}) \subset \mathsf{f}^{-1}(\neg\mathsf{f}(\mathsf{T})) = \neg\mathsf{f}^{-1}(\mathsf{f}(\mathsf{T})) \subset \neg\mathsf{T}$, we have $x \in -_\tau\mathsf{T}$, and hence there exists $\mathsf{V} \in \tau$ such that $x \in \mathsf{V} \subset \neg\mathsf{T} = \neg\neg\mathsf{f}^{-1}(\mathsf{U})$. Thus $-_{\tau_\mathsf{f}} \preceq -_\tau$, by Lemma 2.2.7. Hence there exists an open base σ such that $\tau_\mathsf{f} \sqsubseteq \sigma$ and $-_\tau \simeq -_\sigma$, by Proposition 2.2.8. ■

Corollary 2.2.12 *If* $\mathsf{f} : (\mathsf{X}, \tau) \to (\mathsf{Y}, \tau')$ *is continuous, then* $\mathsf{f} : \langle \mathsf{X}, -_\tau \rangle \to \langle \mathsf{Y}, -_{\tau'} \rangle$ *is continuous.*

Proof. This follows directly from the preceding Theorem 2.2.11. ■

2.3 From Quasi–apartness to Neighbourhood

In this section, we follow Ishihara *et al.* [25] to show how each quasi–apartness space determines two neighbourhood structures on it by using the strongest and the weakest neighbourhood structures. This is of special interest we could see the interplay between these structures. Furthermore, the arguments in this section invoke the powerset axiom whereas the arguments in section 2.2 do not, and we use impredicative constructions to induce the neighbourhood structures on a quasi–apartness space whereas we constructed the quasi–apartness structure on a neighbourhood space in a predicative way.

Proposition 2.3.1 *Let* $\langle X, -\rangle$ *be a quasi–apartness space, and let* $\tau_{-}^{w} := \{-S \mid S \subset X\}$. *Then* (X, τ_{-}^{w}) *is a neighbourhood space.*

Proof. We verify the axioms.

NS1: Let $x \in -S$. Then $x \in -S \subset \neg S$, by QA2, and $-S \in \tau_{-}^{w}$, and hence there exists $U \in \tau_{-}^{w}$ such that $x \in U \subset \neg S$. Since $-S := (\neg S)^{\circ} \subset \neg S$, we have $x \in U \subset -_{\tau_{-}^{w}} S$. Thus $x \in -_{\tau_{-}^{w}} S$.

NS2: Let $x \in -S$. Then $x \in -S \subset \neg S$, by QA2, and $-S \in \tau_{-}^{w}$, and for all $U, V \in \tau_{-}^{w}$ and $x \in U \cap V$ such that $x \in (U \cap V) \subset \neg S$, and hence there exists $W \in \tau_{-}^{w}$ such that $x \in W \subset U \cap V \subset -_{\tau_{-}^{w}} S$. Thus $x \in -_{\tau_{-}^{w}} S$. ■

Proposition 2.3.2 *Let* $-$ *be a quasi–apartness on a set* X. *Then* $- \simeq -_{\tau_{-}^{w}}$.

Proof. Suppose that $x \in -S$. Then $x \in -S \subset \neg S$, by QA2, and $-S \in \tau_{-}^{w}$, and hence $x \in -_{\tau_{-}^{w}} S$. Therefore $- \preceq -_{\tau_{-}^{w}}$.

Conversely suppose that $x \in -_{\tau_{-}^{w}} S$. Then there exists $-T \in \tau_{-}^{w}$ such that $x \in -T \subset \neg S \to -T \subset -S$, by QA4, and so $x \in -S$. Therefore $-_{\tau_{-}^{w}} \preceq -$. Thus $- \simeq -_{\tau_{-}^{w}}$. ■

Proposition 2.3.3 *Let* σ *be an open base on a set* X*. Then* $\tau^{w}_{-\sigma} \sqsubseteq \sigma$*.*

Proof. First note that, $-_{\sigma} \simeq -_{\tau^{w}_{-\sigma}}$, by Proposition 2.3.2. Suppose that $-_{\sigma}S \in \tau^{w}_{-\sigma}$, and let $x \in -_{\sigma}S$. Then there exists $U \in \sigma$ such that $x \in U \subset \neg S$, and therefore since $U \subset -_{\sigma}S$, we have $x \in U \subset -_{\sigma}S$. Thus $\tau^{w}_{-} \sqsubseteq \sigma$. ■

Proposition 2.3.4 *Let* (X, σ) *be a neighbourhood space. Then*

$$\tau^{w}_{-\sigma} \approx \{(\neg\neg U)^{\circ} \mid U \in \sigma\}.$$

Proof. Suppose that $-_{\sigma}S \in \tau^{w}_{-\sigma} \sqsubseteq \sigma$, by Proposition 2.3.3, and let $x \in -_{\sigma}S$. Then there exists $U \in \sigma$ such that $x \in U \subset \neg S$. Since $U \subset -\neg U := (\neg\neg U)^{\circ} \subset \neg\neg U$, we have $x \in U \subset (\neg\neg U)^{\circ} \subset \neg\neg U \subset \neg S$. Therefore $x \in -_{\sigma}S = (\neg S)^{\circ}$, and $-_{\sigma}S \in \{(\neg\neg U)^{\circ} \mid U \in \sigma\}$. Thus

$$\tau^{w}_{-\sigma} \sqsubseteq \{(\neg\neg U)^{\circ} \mid U \in \sigma\}.$$

Conversely suppose that $\{(\neg\neg U)^{\circ} \mid U \in \sigma\}$ is an open base on a set X, and let $x \in (\neg\neg U)^{\circ}$. Then there exists $U \in \sigma$ such that $x \in U \subset \neg\neg U$. Since $-\neg U := (\neg\neg U)^{\circ}$, we have $x \in U \subset -_{\sigma}\neg U \subset \neg S$, and $-_{\sigma}\neg U \in \tau^{w}_{-\sigma}$. Thus

$$\{(\neg\neg U)^{\circ} \mid U \in \sigma\} \sqsubseteq \tau^{w}_{-\sigma}.$$

■

Corollary 2.3.5 *Let* (X, σ) *be a neighbourhood space. Then* $\tau^{w}_{-\sigma} \approx \sigma$ *if and only if*

$$\forall x \in X \forall U \in \sigma[x \in U \rightarrow \exists V \in \sigma(x \in V \wedge (\neg\neg V)^{\circ} \subset U)].$$

Proof. This follows immediately from Proposition 2.3.4. ■

Proposition 2.3.6 *Let* $\langle X, -\rangle$ *be a quasi-apartness space and let* $\tau^{s}_{-} := \{U \subset X \mid U \subset -\neg U\}$*. Then* τ^{s}_{-} *is an open base on* X*.*

Proof. We verify the axioms.

NS1: Suppose that $\langle X, - \rangle$ is a quasi-apartness space. Then there exists $U \in \tau^s_-$ such that $x \in U \subset \neg\emptyset$. Since $-\emptyset = (\neg\emptyset)^\circ \subset \neg\emptyset$, we have $x \in U \subset -_{\tau^s_-}\emptyset$. Therefore $-_{\tau^s_-}\emptyset = X \in \tau^s_-$, by QA1.

NS2: Suppose that $\langle X, - \rangle$ is a quasi-apartness space. Then for each $U, V \in \tau^s_-$ such that $x \in U \cap V$, and hence there exists $W \in \tau^s_-$ such that $x \in W \subset U \cap V$. Since $U \cap V \subset -\neg U \cap -\neg V = -(\neg U \cup \neg V) \subset \neg(\neg U \cup \neg V) = \neg\neg U \cap \neg\neg V = \neg\neg(U \cap V)$, we have $x \in W \subset U \cap V \subset -(\neg U \cup \neg V) \subset \neg\neg(U \cap V)$. Therefore $x \in U \cap V \subset -_{\tau^s_-}\neg(U \cap V)$, by QA4, and hence $U \cap V \in \tau^s_-$. ■

Proposition 2.3.7 *Let* $-$ *be a quasi-apartness on a set* X. *Then* $- \simeq -_{\tau^s_-}$.

Proof. Suppose that $x \in -S$. Then $x \in -S \subset \neg S$, by QA2, and since by Lemma 2.2.3, $-S := -\neg - S$, and hence $x \in -\neg - S \subset \neg S$ and $-\neg - S \in \tau^s_-$. Therefore $- \preceq -_{\tau^s_-}$.

Conversely suppose that $x \in -_{\tau^s_-} S$. Then there exists $U \in \tau^s_-$ such that $x \in U \subset \neg S$, and therefore, since $S \subset \neg U$ implies $-\neg U \subset -S$ and, we have $x \in U \subset -\neg U \subset -S$. Thus $-_{\tau^s_-} \preceq -$. ■

Proposition 2.3.8 *Let* σ *be an open base on a set* X. *Then* $\sigma \sqsubseteq \tau^s_{-_\sigma}$.

Proof. : Suppose that σ be an open base on a set X. Let $U \in \sigma$ be such that $x \in U$. Since $U \subset \neg\neg U$, $U \subset -_\sigma \neg U$, and hence $U \in \tau^s_{-_\sigma}$. Therefore $\sigma \sqsubseteq \tau^s_{-_\sigma}$. ■

Proposition 2.3.9 *Let* (X, σ) *be a neighbourhood space. Then*

$$\tau^s_{-_\sigma} \approx \{S \subset X \mid S \subset (\neg\neg S)^\circ\}.$$

Proof. First notice that $\{X \subset X | S \subset (\neg\neg S)^\circ\}$ is an open base. Because $(\neg\neg S)^\circ = -_\sigma \neg S$, we have

$$\{S \subset X | S \subset (\neg\neg S)^\circ\} = \tau^s_{-_\sigma}.$$

■

Corollary 2.3.10 *Let* (X, σ) *be a neighbourhood space. Then* $\sigma \approx \tau^s_{-_\sigma}$ *if and only if* $S \subset X$ *is open whenever* $S \subset (-\neg S)^\circ$.

Proof. This is clear from Proposition 2.3.9. ■

Proposition 2.3.11 *Let* σ *be an open base on a set* X*. Then* $- \simeq -_\sigma$ *if and only if* $\tau^w_- \sqsubseteq \sigma \sqsubseteq \tau^s_-$.

Proof. Suppose that $- \simeq -_\sigma$. Then it follows from Propositions 2.3.3 and 2.3.8 that

$$\tau^w_{-_\sigma} = \tau^w_- \sqsubseteq \sigma \sqsubseteq \tau^s_- = \tau^s_{-_\sigma}.$$

Conversely, suppose $\tau^w_- \sqsubseteq \sigma \sqsubseteq \tau^s_-$. Then, by Proposition 2.2.2, Lemma 2.2.6, and Proposition 2.3.7,

$$- \simeq -_{\tau^w_-} \preceq -_\sigma \preceq -_{\tau^s_-} \simeq -.$$

■

Theorem 2.3.12 *Let* f *be a function between quasi-apartness spaces* $\langle X, - \rangle$ *and* $\langle Y, -' \rangle$*. Then* $f : \langle X, - \rangle \to \langle Y, -' \rangle$ *is continuous if and only if* $f : (X, \tau^s_-) \to (Y, \tau^s_{-'})$ *is continuous.*

Proof. Suppose that $f : \langle X, - \rangle \to \langle X, -' \rangle$ is continuous, and let $f(x) \in U \in \tau^s_{-'}$. For $y \in f^{-1}(U)$, let $T := f^{-1}(\neg U)$. Since $f(T) \subset \neg U$, we have $f(y) \in U \subset -'\neg U \subset -'f(T)$, and hence $y \in -T = -\neg f^{-1}(U)$. Thus $x \in f^{-1}(U) \subset -\neg f^{-1}(U)$, and so $f^{-1}(U) \in \tau^s_-$.

Conversely suppose that $f : (X, \tau^s_-) \to (Y, \tau^s_{-'})$ is continuous, and let $f(x) \in -'f(S)$. Then, since $-'f(S) = -'\neg -' f(S)$, by Lemma 2.2.3, we have $-'f(S) \in \tau^s_{-'}$, and hence there exists $U \in \tau^s_-$ such that $x \in U \subset f^{-1}(-'f(S))$. Since

$$U \subset f^{-1}(-'f(S)) \subset f^{-1}(\neg f(S)) \subset \neg S,$$

we have $S \subset \neg U$, and hence $x \in U \subset -\neg U \subset -S$. ■

Let $\langle X, - \rangle$ be a quasi–apartness space, and let σ be an open base on X. Then, by Proposition 2.3.11, $- \simeq -_\sigma$ if and only if $\tau^w_- \sqsubseteq \sigma \sqsubseteq \tau^s_-$. Classically, we have $\tau^w_- \approx \tau^s_-$. To see this, if $U \in \tau^s_-$, then $U \subset -\neg U$, therefore, since $U \subset -\neg U \subset \neg\neg U = U$ by classical logic, we have $U \in \tau^w_-$. But the following example due to [25] shows that we cannot prove $\tau^w_- \approx \tau^s_-$ constructively.

Example 2.3.13 Let $P(x)$ be a predicate on a set X, and define an equality $=$ on $X \times \{0, 1\}$ by $(x, 0) = (x, 1) \iff P(x) \vee \neg P(x)$. Then the open base $\sigma := \{\{z\} \mid z \in X \times \{0, 1\}\}$ induces a quasi–apartness $-_\sigma$ on $X \times \{0, 1\}$. It is straightforward to see that $-_\sigma S = \neg S$ and $\sigma \approx \tau^s_{-_\sigma}$. Suppose that $\sigma \sqsubseteq \tau^w_{-_\sigma}$ and let $x \in X$. Then, since $(x, 0) \in \{(x, 0)\} \in \sigma$, there exists $S \subset X \times \{0, 1\}$ such that $(0, 1) \in \neg S \subset \{(x, 0)\}$, and therefore, since $\{(x, 0)\} \subset \neg S$, we have $S \subset \neg\{(x, 0)\}$. Hence $\neg\neg\{(x, 0)\} \subset \neg S \subset \{(x, 0)\}$. If $(x, 1) \in \neg\{(x, 0)\}$, then assuming $P(x) \vee \neg P(x)$, we have $(x, 0) \in \{(x, 0)\}$, a contradiction, and hence $\neg(P(x) \vee \neg P(x))$. Thus

$$\forall x \in X(P(x) \vee \neg P(x)).$$

☺

2.4 Some Interesting Relations

In this section, we follow the presentation given by Ishihara *et al.* in [25] to point out some interesting relations to other theories.

2.4.1 Weak Nested Neighbourhoods Property

The following notion of weak nested neighbourhood was introduced by Bridges and Vîţa for apartness spaces [13] to capture the connection establish connections between certain continuity properties.

Definition 2.4.1 A quasi–apartness space $\langle X, - \rangle$ has the *weakly nested neighbourhoods property* if for all $x \in X$ and $S \subset X$,

WNN. $x \in -S \to \exists T \subset X(x \in -T \wedge \neg T \subset -S)$. ▲

Definition 2.4.2 An open base τ on X is *decent* if

DOB. $\forall x \in X \forall U \in \tau\, [x \in U \to \exists V \in \tau(x \in V \wedge \neg\neg V \subset U)]$. ▲

Proposition 2.4.3 *Let* τ *be a* **DOB** *on a set* X. *Then* $\langle X, -_\tau \rangle$ *has the* **WNN**.

Proof. Suppose that τ be a decent open base on a set X, and let $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$, and hence there exists $V \in \tau$ such that $x \in V \wedge \neg\neg V \subset U \subset \neg S$. Letting $T := \neg V$, we have $\neg T = \neg\neg V \subset U \subset -_\tau S$. Therefore, since $x \in V \subset -_\tau \neg V$, we have $x \in -_\tau T$. ■

Proposition 2.4.4 *Let* σ *be a* **DOB** *on a set* X.*Then* $\tau^w_{-_\sigma} \approx \sigma$.

Proof. This is courtesy of Corollary 2.3.5. ■

Theorem 2.4.5 *Let* f *be a function between neighbourhood spaces* (X, τ) *and* (Y, τ') *such that* τ' *is decent. Then* $f : (X, \tau) \to (Y, \tau')$ *is continuous if and only if* $f : \langle X, -_\tau \rangle \to \langle Y, -_{\tau'} \rangle$ *is continuous.*

Proof. Suppose that $f : (X, \tau) \to (Y, \tau')$ is continuous. Then, by Corollary 2.2.12, $f : \langle X, -_\tau \rangle \to \langle Y, -_{\tau'} \rangle$ is continuous.

Conversely, suppose that $f : \langle X, -_\tau \rangle \to \langle Y, -_{\tau'} \rangle$ is continuous and let $f(x) \in U \in \tau'$. Then there exists $V \in \tau'$ such that $f(x) \in V \wedge \neg\neg V \subset U$. Letting $S := f^{-1}(\neg V)$, we have $f(S) \subset \neg V$, and so $f(x) \in V \subset \neg\neg V \subset \neg f(S)$. Hence $f(x) \in -_{\tau'} f(S)$, and therefore $x \in -_\tau S$. Thus there exists $W \in \tau$ such that

$$x \in W \subset \neg S = \neg f^{-1}(\neg V) = f^{-1}(\neg\neg V) \subset f^{-1}(U).$$

Thus $f : (X, \tau) \to (Y, \tau')$ is also continuous. ■

Proposition 2.4.6 *Let* $\langle X, -\rangle$ *be a quasi–apartness space. Then* $\langle X, -_{\tau}\rangle$ *has the* **WNN** *if and only if* τ_{-}^{w} *is a* **DOB**.

Proof. Suppose that $\langle X, -_{\tau}\rangle$ has the weakly nested neighbourhoods property, and let $x \in -S \in \tau_{-}^{w}$. Then there exists $-T \in \tau_{-}^{w}$ such that $x \in -T \wedge \neg T \subset -S$. Since $-T = -\neg - T \subset \neg\neg - T \rightarrow -T \subset \neg\neg\neg T$, by Lemma 2.2.3 and QA2, and $\neg\neg\neg T := \neg T$, we have $x \in -T \wedge \neg\neg - T \subset \neg\neg\neg T = \neg T \subset -S$.

Conversely suppose that τ_{-}^{w} is decent, and let $x \in -S$. Then, since $-S \in \tau_{-}^{w}$, there exists $-T' \in \tau_{-}^{w}$ such that $x \in -T' \wedge \neg\neg - T' \subset -S$. Since we letting $T := \neg - T'$, we have $x \in -T' \wedge \neg T \subset -S$. Therefore, since $-T = -\neg - T' = -T'$ by Lemma 2.2.3 and $\neg T = \neg\neg - T' \subset -S$, we have $x \in -T' \wedge \neg\neg - T' \subset -S$. Therefore $x \in -T \wedge \neg T \subset -S$. ■

Theorem 2.4.7 *Let* f *be a function between quasi–apartness spaces* $\langle X, -\rangle$ *and* $\langle X, -'\rangle$ *such that* $\langle Y, -'\rangle$ *has the* **WNN** *property. Then* $f : \langle X, -\rangle \rightarrow \langle Y, -'\rangle$ *is continuous if and only if* $f : (X, \tau_{-}^{w}) \rightarrow (Y, \tau_{-'}^{w})$ *is continuous.*

Proof. : Note first that $- \simeq -_{\tau_{-}^{w}}$ and $-' \simeq -_{\tau_{-'}^{w}}$, by Proposition 2.3.2. Thus $f : \langle X, -\rangle \rightarrow \langle Y, -'\rangle$ is continuous. Furthermore, by Proposition 2.4.6, $\tau_{-'}^{w}$ is a **DOB** and, therefore, by Proposition 2.4.5, $f : \left(X, -_{\tau_{-}^{w}}\right) \rightarrow \left(Y, -_{\tau_{-'}^{w}}\right)$ is continuous if and only if $f : (X, \tau_{-}^{w}) \rightarrow \left(Y, \tau_{-'}^{w}\right)$ is continuous. ■

2.4.2 Spaces with Inequality

We now turn the exposition to what alluded to earlier where Richman in [32] showed that the inequality required in the definition of an apartness space is actually determined by the point–set apartness on the set; so, it can be viewed as a derivable concept. However, as Ishihara *et al.* pointed out in [25] that this is not the case for quasi–apartness spaces. To see this, consider the quasi–apartness space induced from the neighbourhood space $X = \{0, 1\}$ with $0 \neq 1$ and the open base $\tau = \{X\}$. In this case the derivable inequality

$$x \neq' y \quad \text{meaning} \quad x \in -_\tau\{y\} \vee y \in -_\tau\{x\}$$

satisfies $\forall x, y \in X \neg(x \neq' y)$.

So the following explains how inequality and quasi–apartness are related.

Definition 2.4.8 A *weak apartness space* $\langle X, -, \neq\rangle$ is a quasi–apartness space $\langle X, -\rangle$ with the inequality $\neq$ satisfying the axiom A2. ▲

Note that A2 implies QA2 as we have seen in Proposition 2.2.1. Since $\sim S \subset \neg S$ and by A2, $-S \subset \sim S$, we have QA2, $-S \subset \neg S$.

Definition 2.4.9 A *neighbourhood space with an inequality* $\neq$ is a neighbourhood space (X, τ) satisfying for all $x, y \in X$,

T_0^{-1}. $\qquad \exists U \in \tau[(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)] \to x \neq y$. ▲

Lemma 2.4.10 *Let* (X, τ) *be a neighbourhood space with an inequality* $\neq$*. Then* $x \in -_\tau S$ *if and only if there exists* $U \in \tau$ *such that* $x \in U \subset\sim S$*.*

Proof. Suppose that $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset\sim S$.

Conversely, suppose that there exists $U \in \tau$ such that $x \in U \subset\sim S$, and let $x \in -S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$. Since $S \subset \neg U$, we have for all $z \in U$ and $y \in S$, $z \neq y$, by T_0^{-1}, and hence $x \in U \subset\sim S$. ■

Proposition 2.4.11 *Let* (X, τ) *be a neighbourhood space with an inequality* $\neq$*. Then* $\langle X, -, \neq\rangle$ *is a weak apartness space.*

Proof. We need to show that A2 holds. Let $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset\sim S$, by Lemma 2.4.10. ■

Proposition 2.4.12 *Let* $\langle X, -, \neq \rangle$ *be a weak apartness space. Then* (X, τ_{-}^{s}) *is a neighbourhood space with the inequality* $\neq$*.*

Proof. Suppose that there exists $U \in \tau_{-}^{s}$ such that

$$x \in U \wedge y \notin U \quad \text{or} \quad x \notin U \wedge y \in U.$$

In the former case, $x \in U \subset -\neg U \subset\sim \neg U$, by A2, and hence $x \neq y$. In the latter case, $y \in U \subset -\neg U \subset\sim \neg U$, by A2 again, and hence $x \neq y$. ■

Corollary 2.4.13 *Let* $\langle X, -, \neq \rangle$ *be a weak apartness space, and let* τ *be an open base on* X *with* $- \simeq -_{\tau}$*. Then* (X, τ) *is a neighbourhood space with the inequality* $\neq$*.*

Proof. Suppose that there exists $U \in \tau$ such that

$$x \in U \wedge y \notin U \quad \text{or} \quad x \notin U \wedge y \in U.$$

In the former case, since $\tau \sqsubseteq \tau_{-}^{s}$, by Proposition 2.3.11, there exists $V \in \tau_{-}^{s}$ such that $x \in V \subset U$, and hence $y \notin V$. Therefore $x \neq y$, by Proposition 2.4.12. Similarly in the latter case, since $\tau \sqsubseteq \tau_{-}^{s}$, by Proposition 2.3.11, there exists $V \in \tau_{-}^{s}$ such that $y \in V$ and hence $x \notin V$. Therefore $x \neq y$, by Proposition 2.4.12. ■

In light of the work of Ishihara *et al.* in [25], it is more natural to remove the requirement that the inequality be nontrivial which is refined in the next definition of an apartness space.

Definition 2.4.14 An *apartness space* is a weak apartness space $\langle X, -, \neq \rangle$ satisfying the axioms A1 and A5. ▲

We recall he notion of separated space which was introduced by Grayson in [18].

Definition 2.4.15 A separated space is a neighbourhood space (X, τ) with an inequality $\neq$ satisfying

SEP1. $\forall x \in X(\sim\{x\}$ is open$)$,

SEP2. $\forall x \in X \forall U \in \tau[x \in U \to \forall y \in X(x \neq y \vee y \in U)]$. ▲

It is shown in [18], Grayson showed that SEP2 implies T_0^{-1}.

Proposition 2.4.16 *Let* (X, τ) *be a separated space with an inequality* $\neq$. *Then* $\langle X, -_\tau, \neq \rangle$ *is an apartness space.*

Proof. We verify the two axioms.

A1. Suppose that $x \neq y$. Then $y \in \sim\{x\}$, and therefore, since $\sim\{x\}$ is open, by SEP1, there exists $U \in \tau$ such that $y \in U \subset \sim\{x\}$. Hence $y \in -_\tau\{x\}$, by Lemma 2.4.10.

A5. Suppose that $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$, and hence for each $y \in X$ either $x \neq y$ or $y \in U$, by SEP2. In the latter case, since $U \subset -_\tau S$, we have $y \in -_\tau S$. ■

Proposition 2.4.17 *Let* $\langle X, -, \neq \rangle$ *be an apartness space. Then* (X, τ_-^w) *is a separated space with the inequality* $\neq$.

Proof. We verify the two axioms.

SEP1. Suppose that $y \in \sim\{x\}$. Then $x \neq y$, and hence $y \in -\{x\} \subset \sim\{x\}$, by A1 and A2. Therefore $\sim\{x\}$ is open in (X, τ_-^w).

SEP2. Suppose that $x \in -S \in \tau_-^w$. Then for each $y \in X$ either $x \neq y$ or $y \in -S$, by A5. ■

Chapter 3

Separation Properties in Quasi–Apartness Spaces and Neighbourhood Spaces

3.1 Introduction

In this chapter, we follow the work of Havea *et al.* in [21] to investigate the separation properties of neighbourhood spaces in the framework of constructive mathematics in light of [3]. We also show that these separation properties can be carried over into the context of induced and product spaces as in classical mathematics. Furthermore, we define the corresponding separation properties for quasi–apartness spaces, and show that the T_i^+ separation properties have some advantage over the T_i separation properties, and, finally, we deal with separation properties for spaces with inequality.

As mentioned in Chapter 2, Ishihara *et al.* [25] showed that the notion of quasi–apartness space is more general than the notion of a (point–set) apartness space in the sense of of Bridges and Vîţă [12], and there is an adjunction between the category of neighbourhood spaces and the category of quasi–apartness spaces. Constructions of limits and colimits in the category of neighbourhood spaces can be carried over to the category of quasi–apartness spaces under the adjunction [25].

3.2 Separation Axioms

Definition 3.2.1 Let (X, τ) be a neighbourhood space. Then the *the interior* of $S \subseteq X$ is

$$S^\circ := \{x \in S \mid (\exists U \in \tau)(x \in U \subset S)\} \subseteq X.$$

We say that S is *open* if $S = S^\circ$. ▲

As we saw in Chapter 2, we can naturally associate with the neighbourhood spaces (X, τ), a quasi–apartness $-_\tau$ by letting $-_\tau S = (\neg S)^\circ$.

Definition 3.2.2 We see that an open base τ on a set X is *compatible with a quasi-apartness* $-$ *on* X if the induced quasi–apartness $-$ and the original one $-$ are equivalent, that is, if for every subset S of X we have $-_\tau S = -S$. ▲

Definition 3.2.3 Let σ and τ be two open bases on a set X. Then we say that σ *is weaker than* τ (or τ *is stronger than* σ) if for each $x \in X$ and $U \in \sigma$ with $x \in U$ there exists $V \in \tau$ such that $x \in V \subset U$. ▲

Ishihara *et al.* [25] showed that if $\langle X, -\rangle$ is a quasi–apartness space, then

$$\tau_-^w := \{-S \mid S \subset X\} \quad \text{and} \quad \tau_-^s := \{U \subset X \mid U \subset -\neg U\}$$

are open bases on X. Moreover, τ_-^w is the weakest open base on X compatible with $-$, and τ_-^s is the strongest open base on X compatible with $-$.

Let (X, τ) be a neighbourhood space. Grayson in [18] defined a binary relation

$\sim_i$ for $i = 0, 1, 2$ on X in the following manner.

$$\begin{aligned} x \sim_0 y &:= (\forall U \in \tau)[x \in U \Longleftrightarrow y \in U] \\ x \sim_1 y &:= (\forall U \in \tau)[x \in U \Longrightarrow y \in U] \\ x \sim_2 y &:= (\forall U, V \in \tau)[x \in U \wedge y \in V \Longrightarrow \exists z(z \in U \cap V)] \end{aligned}$$

Furthermore, Grayson introduced the constructive formulations of the T_i separation properties: in a neighbourhood space X,

$$\mathrm{T}_i := (\forall x, y \in X)\, [x \sim_i y \Longrightarrow x = y] \qquad (i = 0, 1, 2).$$

Recently, Aczel and Fox [2] introduced the binary relations $\neq_i$, for all $i = 0, 1, 2$, on a constructive topological space[1]. The corresponding relations for a neighbourhood space (X, τ) is defined below.

$$\begin{aligned} &\mathrm{T}_0 \quad (x \neq_0 y) := (\exists U \in \tau)[(x \in U \wedge \neg(y \in U)) \vee (\neg(x \in U) \wedge y \in U)], \\ &\mathrm{T}_1 \quad (x \neq_1 y) := (\exists U \in \tau)[(x \in U \wedge \neg(y \in U))], \\ &\mathrm{T}_2 \quad (x \neq_2 y) := (\exists U, V \in \tau)[x \in U \wedge y \in U \wedge U \cap V = \emptyset]. \end{aligned}$$

The T_i^+ separation properties are defined as follow. In a neighbourhood space X,

$$\mathrm{T}_i^+ := (x, y \in X)\, [\neg\, (x \neq_i y) \Longrightarrow x = y] \qquad (i = 0, 1, 2).$$

In particular:

$$\begin{aligned} &\mathrm{T}_0^+ \quad \neg\, (x \neq_0 y) := (\forall U \in \tau)[\neg(x \in U) \Longleftrightarrow \neg(y \in U)], \\ &\mathrm{T}_1^+ \quad \neg\, (x \neq_1 y) := (\forall U \in \tau)[\neg(y \in U) \Longrightarrow \neg(x \in U)], \\ &\mathrm{T}_2^+ \quad \neg\, (x \neq_2 y) := (\forall U, V \in \tau)[x \in U \wedge y \in V \Longrightarrow \neg(U \cap V = \emptyset)]. \end{aligned}$$

---

[1] The notion of a constructive topological space is general enough to include the spaces of formal points of a formal topology [21].

As tabulated in [21], we have the following implications among the separation axioms.

$$\begin{tikzcd}
T_2^+ \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_1^+ \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_0^+ \arrow[d, Rightarrow] \\
T_2 \arrow[r, Rightarrow] & T_1 \arrow[r, Rightarrow] & T_0
\end{tikzcd}$$

3.3 Separation Properties

Definition 3.3.1 Let(X, τ) be a neighbourhood space. Then *the closure* $\overline{S}$ of a subset S of X is the set

$$\overline{S} := \{x \in X \mid (\forall U \in \tau)[x \in U \Longrightarrow \exists y(y \in U \cap S)]\},$$

and a subset S of X is *closed* if $S := \overline{S}$. A subset S of X is *strongly closed* if there exists an open set $T \subset X$ such that $S := \neg T$. ▲

Definition 3.3.2 A mapping f from a neighbourhood space (X, τ) into a neighbourhood space (Y, σ), $f : (X, \tau) \to (X, \sigma)$, is *continuous* if $f^{-1}(U)$ is open in X for each $U \in \sigma$. ▲

Note that the inverse image of each closed (strongly closed) set under a continuous mapping is closed (strongly closed, respectively).

Definition 3.3.3 Let X be a set, let $\{(Y_i, \sigma) \mid i \in I\}$ be a family of neighbourhood spaces, and for each $i \in I$ let f_i be a mapping of X into Y_i. Then *the initial topological structure* τ *on* X *for the family* $\{f_i \mid i \in I\}$ defined by

$$\tau := \left\{f^{-1}(U_{i_1}) \cap \cdots \cap f^{-1}(U_{i_n}) \mid i_k \in I \wedge U_{i_k} \in \sigma_{i_k} \text{ for each } k = 1, \ldots, n\right\}$$

is the weakest open base on X for which the mappings f_i are continuous [7]. ▲

It is worth mentioning that the product of neighbourhood spaces (X, τ) and (Y, σ) is the set $X \times Y$ with the initial topological structure for the projections $pr_X : X \times Y \to X$ and $pr_Y : X \times Y \to Y$.

The next two results give characterisations for T_1 and T_1^+ spaces.

Proposition 3.3.4 *A neighbourhood space* (X, τ) *is* T_1 *if and only if* $\{x\}$ *is closed for each* $x \in X$.

Proof. Since (X, τ) is T_1,

$$\begin{aligned}
& (\exists U \in \tau)[x \in U \wedge \neg(y \in U)] \quad \forall x, y \in X \\
\Longleftrightarrow\ & (\exists U \in \tau)[x \in U \subset \neg\{y\} \Longrightarrow x \in U \subset -_\tau\{y\}] \quad \text{(by QA2)} \\
\Longleftrightarrow\ & (\exists U \in \tau)[x \in U \Longrightarrow \exists x'(x' \in U)] \\
\Longleftrightarrow\ & \forall U \in \tau)[x \in U \Longrightarrow x' \in U] \text{ for } x \sim_2 x' \\
\Longleftrightarrow\ & (\forall U \in \tau)[x \in U \Longrightarrow x' \in U \subset \{x\}] \\
\Longleftrightarrow\ & (\forall U \in \tau)[x \in U \Longrightarrow x' \in U \cap \{x\}] \text{ for } U \cap \{x\} = U.
\end{aligned}$$

■

Proposition 3.3.5 *A neighbourhood space* (X, τ) *is* T_1^+ *if and only if* $\{x\}$ *is strongly closed for each* $x \in X$.

Proof. Since $\{x\}$ is strongly closed for each $x \in X$,

$$\begin{aligned}
& (\forall U \in \tau)[x \in U \Longrightarrow \exists y(y \in U \cap \{x\})] \\
\Longleftrightarrow\ & (\forall U \in \tau)[x \in U \subset \{y\} \Longrightarrow y \in U \subset \{x\}] \\
\Longleftrightarrow\ & (\forall U \in \tau)[x \in U \Longrightarrow y \in U] \text{ for } x \sim_2 y \\
\Longleftrightarrow\ & (\forall V \in \tau)[\neg(y \in V) \Longrightarrow \neg(x \in V)] \text{ by contrapositive, and } U \cap V = \emptyset.
\end{aligned}$$

■

Proposition 3.3.6 *Let* (X, τ) *be a neighbourhood space. Then the following are equivalent conditions.*

(i) (X, τ) *is* T_2.

(ii) *The diagonal* $\Delta_X := \{(x, x) \mid x \in X\}$ *is closed in* $X \times X$.

(iii) *The graph of any continuous mapping from a neighbourhood space* (Z, σ) *into* (X, τ) *is closed in* $Z \times X$.

Proof. The equivalence of (i) and (ii) is easily established since

$$\begin{aligned}(x, y) \in \overline{\Delta_X} \quad &\Leftrightarrow \quad (\forall U, V \in \tau)[(x, y) \in U \times V \Rightarrow \exists \xi (\xi \in U \times V \cap \Delta x)] \\ &\Leftrightarrow \quad (\forall U, V \in \tau)[x \in U \wedge y \in V \Rightarrow \exists z (z \in U \cap V)] \\ &\Leftrightarrow \quad x \sim_2 y.\end{aligned}$$

Suppose Δ_X is closed, and let f be a continuous mapping from (Z, σ) into (X, τ). Then the mapping $F : Z \times Y \rightarrow X \times Y$, defined by

$$F(x, y) := (f(x), y),$$

is clearly continuous and hence the graph $F^{-1}(\Delta_X)$ of f is closed since Δ_X is closed. Thus (ii) implies (iii).

Lastly, notice that Δ_X is the graph of the identity map on X. Hence Δ_X is closed. Thus (iii) implies (ii). ■

Our next result is a slight variation of the preceding one.

Proposition 3.3.7 *Let* (X, τ) *be a neighbourhood space. Then the following are equivalent statements.*

(i) (X, τ) *is* T_2^+.

(ii) Δ_X *is strongly closed in* $X \times X$.

(iii) *The graph of any continuous mapping from a neighbourhood space* (Z, σ) *into* (X, τ) *is strongly closed in* $Z \times X$.

Proof. Let

$$\mathsf{T} := \bigcup \{\mathsf{U} \times \mathsf{V} \in \tau \times \tau | \mathsf{U} \cap \mathsf{V} = \emptyset\}.$$

It is easy to see that T is open, $\Delta_X \subset \neg \mathsf{T}$, and that $(x, y) \in \mathsf{T}$ if and only if $x \neq_2 y$. Suppose (i). Then

$$(x, y) \in \neg \mathsf{T} \;\Leftrightarrow\; \neg (x \neq_2 y) \;\Rightarrow\; x = y \;\Leftrightarrow\; (x, y) \in \Delta_X$$

and hence $\Delta_X = \neg \mathsf{T}$. Thus (i) implies (ii).

Suppose that $\Delta_X = \neg S$ for some open set S, and let (x, y) be an element of S. Then there exist $U, V \in \tau$ such that

$$(x, y) \in U \times V \subset S \subset \neg \Delta x \quad \Rightarrow \quad U \cap V = \emptyset;$$

that is, $(x, y) \in \mathsf{T}$. Therefore,

$$\neg (x \neq_2 y) \;\Leftrightarrow\; (x, y) \in \neg \mathsf{T} \;\Rightarrow\; (x, y) \in \neg S \;\Rightarrow\; (x, y) \in \Delta_X \;\Rightarrow\; x = y,$$

and so (X, τ) is T_2^+.

Lastly, the equivalence of (ii) and (iii) can be shown similarly to the proof of Proposition 3.3.6. ■

The next theorem is of special interest because it shows that separation properties can be lifted into the context of topological structure.

Theorem 3.3.8 *Let* X *be a set, let* $\{(Y_j, \sigma_j) \mid j \in J\}$ *be a family of neighbourhood spaces, and for each* $j \in J$ *let* f_j *be a mapping of* X *into* Y_j *such that for all* $x, y \in X$,

$$(\forall j \in J)(f_j(x) = f_j(y)) \Longrightarrow x = y.$$

Then, for $i = 0, 1, 2$, *the initial topological structure on* X *for the family* $\{f_j \mid j \in J\}$ *is* T_i (T_i^+) *if* (Y_j, σ_j) *is* T_i $(\mathrm{T}_i^+$, *respectively) for each* $j \in J$.

Proof. Suppose that X be a set with a τ is an open base on X, and given $\{(\mathsf{Y}_j, \sigma_j) \mid j \in J\}$ be a family of neighbourhood spaces. Let $f_j : (X, \tau) \to (Y_j, \sigma_j)$ for all $j \in J$, and $\neg T := \triangle_X$ is closed in $X \times Y_j$. Let $f_j(x) \neq_i f_{(}y) \in -_{\sigma_j} T$. Then there exists $U, V_j \in \sigma_j$ such that $f_j(x) \neq_i f_j(y) \in V_j \subset \neg f(T) \Longrightarrow x \neq_i y \in U \subset f_j^{-1}(V_j)$ for $i = 0, 1, 2$, and for each $x, y \in X$. Since $x \neq_i y \in U \subset f_j^{-1}(V_j) \subset f_j^{-1}(f_j(T)) = \neg f_j^{-1}(f_j(T)) \subset \neg T$, we have $f_j(x) \neq_i f_j(y) \in V_j \subset \neg f(T) \Longrightarrow x \neq_i y \in U \subset f_j^{-1}(V_j) \subset \neg T := \triangle_X$. Therefore $(\forall j \in J)(f_j(x) \neq_i f_j(y)) \Longrightarrow x \neq_i y)$, for $i = 0, 1, 2$. Thus the initial topological structure on X for the family $\{f_j(x) \mid j \in J\}$ is T_i for $i = 0, 1, 2$.

On the other hand, suppose that X be a set with a τ is an open base on X, and given $\{(Y_j, \sigma_j) \mid j \in J\}$ be a family of neighbourhood spaces. Let $f_j : (X, \tau) \to (Y_j, \sigma_j)$ for all $j \in J$, and $\neg T := \triangle_X$ is closed in $X \times Y_j$. Let $\neg \left(f_j(x) \neq_i f_{(}y)\right) \in -_{\sigma_j} T$. Then there exists $U, V_j \in \sigma_j$ such that $\neg(f_j(x) \neq_i f_{(}y)) \in V_j \subset \neg f(T) \Longrightarrow \neg(x \neq_i y) \in U \subset f_j^{-1}(V_j)$ for $i = 0, 1, 2$, and for each $x, y \in X$. Since $\neg(x \neq_i y) \in U \subset f_j^{-1}(V_j) \subset f_j^{-1}(f(T)) = \neg f_j^{-1}(f_j(T)) \subset \neg T$, we have $\neg(f_j(x) \neq_i f_j(y)) \in V_j \subset \neg f(T) \Longrightarrow \neg(x \neq_i y) \in U \subset f_j^{-1}(V_j) \subset \neg T := \triangle_X$. Therefore, since $(\forall j \in J)[(\neg(f_j(x) \neq_i f_j(y))) \Longrightarrow (f_j(x) \neq_i f_j(y))$, whenever $(\neg(x \neq_i y)) \Longrightarrow x = y]$, for all $i = 0, 1, 2$. Thus the initial topological structure on X for the family $\{f_j(x) \mid j \in J\}$ is T_i^+ for $i = 0, 1, 2$. ■

Corollary 3.3.9 *Let* (Y, σ) *be a neighbourhood space, and let* X *be a subset of* Y. *Then, for* $i = 0, 1, 2$, *the induced topology on* X *is* T_i (T_i^+) *if* (Y, σ) *is* T_i (T_i^+, *respectively*).

Proof. This follows immediately from Theorem 3.3.8 since the induced topology on X is the initial topological structure from the canonical injection from X to Y. ■

Corollary 3.3.10 *Let* $\{(X, \sigma_j) \mid j \in J\}$ *be a family of neighbourhood spaces. Then, for* $i = 0, 1, 2$, *the product topology on* $\prod_{j \in J} X_j$ *is* T_i (T_i^+) *if each neighbourhood space* (X, σ_j) *is* T_i (T_i^+, *respectively*).

Proof. This follows immediately from Theorem 3.3.8 because the product topology on $\prod_{j\in J} X_j$ is in fact the the initial topological structure for the projection pr_j of $\prod_{j\in J} X_j$ onto X_j. ■

Definition 3.3.11 Let $\langle X, -\rangle$ be a quasi–apartness space. Then, for $i = 0, 1, 2$, we say that $\langle X, -\rangle$ *is* T_i (T_i^+) if (X, τ_-^w) is T_i (T_i^+, respectively). ▲

Proposition 3.3.12 *Let* $\langle X, -\rangle$ *be a quasi–apartness space. Then, for* $i = 0, 1, 2$, $\langle X, -\rangle$ *is* T_i (T_i^+) *if and only if* (X, τ) *is* T_i (T_i^+, *respectively*) *for any open base* τ *compatible with* $-$.

Proof. Note that if σ and τ are open bases on a set X such that σ is weaker than τ, then (X, τ) is T_i (T_i^+) whenever (X, σ) is T_i (T_i^+, respectively) for $i = 0, 1, 2$. ■

Proposition 3.3.13 *Let* $\langle X, -\rangle$ *be a quasi-apartness space. Then, for* $i = 0, 1, 2$, $\langle X, -\rangle$ *is* T_i^+ *if* (X, τ) *is* T_i^+ *for some open base* τ *compatible with* $-$.

Proof. Suppose that $x \in U$ and $\neg(y \in U)$ for some $U \in \tau$. Then, since $x \in U \subset \neg\neg U$, we have $x \in -\neg U$, and if $y \in -\neg U$, then $y \in \neg\neg U$, a contradiction; whence $\neg(y \in -\neg U)$. Therefore, for $i = 0, 1$, $x \neq_i y$ in τ implies $x \neq_i y$ in τ_-^w.

Suppose that $x \in U$ and $y \in V$ for some $U, V \in \tau$ with $U \cap V = \emptyset$. Then $x \in -\neg U$ and $y \in -\neg V$. Assume that

$$z \in -\neg U \cap -\neg V \subset \neg\neg U \cap \neg\neg V.$$

If $z \in U$, then $z \in U \cap V$, a contradiction, when $z \in V$, and hence $z \in \neg V$, a contradiction again. Thus $z \in \neg U$, and this contradiction ensures that $-\neg U \cap -\neg V = \emptyset$. Therefore $x \neq_2 y$ in τ implies $x \neq_2 y$ in τ_-^w. ■

3.4 Spaces with Inequality

We close this chapter with careful of analysis of spaces that come equipped with an inequality. We recall the definition of a (point–set) apartness space given in [11, 12] assumes the existence of an inequality relation on the underlying set. But first, we recall the following definitions.

Definition 3.4.1 An *inequality* $\neq$ *on a set* X such that

- $x \neq y \Longrightarrow y \neq x$ and
- $x \neq y \Longrightarrow \neg(x = y)$.

The inequality $\neq$ is *tight* if

- $\neg(x \neq y) \Longrightarrow x = y$. ▲

As it then turns out the given inequality can be determined by the (point–set) apartness on the set [32]. When we deal with a quasi–apartness though, as shown in [25], this is no longer the case.

Definition 3.4.2 For a subset S of a set X with an inequality $\neq$, the *complement of* S is the set

$$\sim S := \{x \in X \mid (\forall y \in S)(x \neq y)\}.$$

▲

Definition 3.4.3 A *weak apartness space* is a quasi–apartness $\langle X, -\rangle$ with an inequality $\neq$ satisfying the following axiom instead of $(Q2)$:

(A2) $-S \subset \sim S$. ▲

For a neighbourhood space with an inequality, we need a compatibility condition between the inequality and the open base.

Definition 3.4.4 A *neighbourhood space with an inequality* $\neq$ is a neighbourhood space (X, τ) satisfying the following condition for all $x, y \in X$:

$(*)$ $x \neq_0 y \Longrightarrow x \neq y$. ▲

Notice that condition $(*)$ is actually the topological equivalent to the axiom $(A2)$ as stipulated in [25].

We now introduce $T_i^{\neq}$ separation properties: for $i = 0, 1, 2$, a neighbourhood space (X, τ) with an inequality $\neq$ is $T_i^{\neq}$ if

$$(\forall x, y \in X)[x \neq y \Longrightarrow x \neq_i y].$$

Note that $T_2^{\neq}$ spaces correspond to Hausdorff spaces [12], and that if the inequality is tight, then the following implications hold as given by Havea *et al.* in [21].

$$\begin{tikzcd}
T_2^{\neq} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_1^{\neq} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_0^{\neq} \arrow[d, Rightarrow] \\
T_2^{+} \arrow[r, Rightarrow] & T_1^{+} \arrow[r, Rightarrow] & T_0^{+}
\end{tikzcd}$$

Proposition 3.4.5 *The following are equivalent conditions.*

(i) *A neighbourhood space* (X, τ) *with an inequality* $\neq$ *is* $T_1^{\neq}$.

(ii) $\sim \{x\}$ *is open for each* $x \in X$.

Proof. Suppose that (X, τ) is $T_1^{\neq}$. Let $y \in \sim \{x\}$. Then $y \neq_1 x$, hence there is $U \in \tau$ such that $\neg(x \in U) \wedge y \in U$. If $z \in U$, then $z \in U \wedge \neg(x \in U)$, and hence $z \neq x$ by $(*)$. Therefore $y = z \in U \subset \sim \{x\}$. This proves (i) implies (ii).

Conversely, suppose that $\sim \{x\}$ is open, and let $y \neq x$. Then $y \in \sim \{x\}$, and hence there exists $U \in \tau$ such that $y \in U \subset \sim \{x\}$. Therefore $y \in U \wedge \neg(x \in U)$, and so $y \neq_1 x$. This proves (ii) implies (i). ■

Proposition 3.4.6 *Let* (X, τ) *be a neighbourhood space with an inequality* $\neq$. *Then the following are equivalent conditions.*

(i) (X, τ) *is* $T_2^{\neq}$.

(ii) $\{(x, y) \in X \times X \mid x \neq y\}$ *is open.*

(iii) *If* f *is a continuous mapping from a neighbourhood space* (Z, σ) *into* (X, τ), *then* $\{(x, y) \in Z \times X \mid y \neq f(x)\}$ *is open.*

Proof. Suppose that (X, τ) is $T_2^{\neq}$. Let $x \neq y$. Then $x \neq_2 y$, hence there exist $U, V \in \tau$ with $U \cap V = \emptyset$ such that $x \in U$ and $y \in V$. If $u \in U$ and $v \in V$, then $u \in U \wedge \neg(v \in U)$, and hence $u \neq v$ by $(*)$. Therefore

$$(u, v) \in U \times V \subset \{(x, y) \in X \times X \mid x \neq y\}.$$

This proves (i) implies (ii)

Conversely, suppose that $\{(x, y) \in X \times X \mid x \neq y\}$ is open. If $x \neq y$, then there exists $U, V \in \tau$ such that $(x, y) \in U \times V \subset \{(u, v) \in X \times X \mid u \neq v\}$, and so $U \cap V = \emptyset$. Thus (X, τ) is $T_2^{\neq}$ is equivalent to the set $\{(x, y) \in X \times X \mid x \neq y\}$ is open. This proves (ii) implies (i).

The equivalence of (ii) and (iii) can be established similar to the proof of Proposition 3.4.6. ■

The next result gives a characterisation of the relation between $T_0^{\neq}$ neighbourhood spaces and weak apartness spaces.

Proposition 3.4.7 *Let* $\langle X, - \rangle$ *be a weak apartness space with an inequality* $\neq$. *If* (X, τ) *is* $T_0^{\neq}$ *for some open base* τ *compatible with* $-$, *then for all* $x, y \in X$,

(A0) $x \neq y \Rightarrow x \in -\{y\} \vee y \in -\{x\}$.

Conversely, if $\langle X, - \rangle$ *satisfies* (A0), *then* (X, τ) *is* $T_0^{\neq}$ *for any open base* τ *compatible with* $-$.

Proof. Let (X, τ) be $T_0^{\neq}$ with τ compatible with $-$, and suppose that $x \neq y$. Then there exists $U \in \tau$ such that either

$$x \in U \wedge \neg(y \in U) \quad \text{or} \quad \neg(x \in U) \wedge y \in U \qquad \text{for some } U \in \tau.$$

In the former case, since $x \in U \subset \neg\{y\}$, we have $x \in -_{\tau}\{y\} = -\{y\}$, by QA2 and Proposition 2.3.11. In the latter case, since $y \in U \subset \neg\{x\}$, we have $y \in -_{\tau}\{x\} = -\{x\}$, by QA2 and Proposition 2.3.11. Therefore $x \neq y \Longrightarrow x \in -_{\tau}\{y\} \vee y \in -_{\tau}\{x\}$.

Conversely, suppose that $\langle X, -\rangle$ satisfies (A0), that τ is an open base on X compatible with $-$, and that $x \neq y$. Then either $x \in -\{y\}$ or $y \in -\{x\}$. In the former case, because $x \in -\{y\} = -_{\tau}\{y\}$, there exists $U \in \tau$ such that $x \in U \subset \neg\{y\}$, and hence $x \in U \wedge \neg(y \in U)$. In the latter case, because $y \in -\{x\} = -_{\tau}\{x\}$, there exists $U \in \tau$ such that $y \in U \subset \neg\{x\}$, and hence $y \in U \wedge \neg(x \in U)$ for some $U \in \tau$. Thus (X, τ) is $T_0^{\neq}$ for some open base τ compatible with $-$. ■

Proposition 3.4.8 *Let $\langle X, -\rangle$ be a weak apartness space with an inequality $\neq$. If (X, τ) is $T_1^{\neq}$ for some open base τ compatible with $-$, then for all $x, y \in X$,*

(A1) $x \neq y \Rightarrow x \in -\{y\}$.

Conversely, if $\langle X, -\rangle$ satisfies (A1), then (X, τ) is $T_1^{\neq}$ for any open base τ compatible with $-$.

Proof. Let (X, τ) be $T_1^{\neq}$ with τ compatible with $-$, and suppose that $x \neq y$. Then $x \in U \wedge \neg(y \in U)$ for some $U \in \tau$. Since $x \in U \subset \neg\{y\}$, we have $x \in -_{\tau}\{y\} = -\{y\}$. Thus $(A1)$.

Conversely, suppose that $\langle X, -\rangle$ satisfies $(A1)$, that τ is an open base on X compatible with $-$, and that $x \neq y$. Then $x \in -\{y\}$. Because $x \in -\{y\} = -_{\tau}\{y\}$, there exists $U \in \tau$ such that $x \in U \subset \neg\{y\}$, and hence $x \in U \wedge \neg(y \in U)$. Thus (X, τ) is $T_1^{\neq}$ for some open base τ compatible with $-$. ■

Proposition 3.4.9 *Let $\langle X, -\rangle$ be a neighbourhood space with an inequality $\neq$. If (X, τ) is $T_2^{\neq}$ for some open base τ compatible with $-$, then (X, τ_-^w) is $T_2^{\neq}$. Conversely, if $\langle X, \tau_-^w\rangle$ is $T_2^{\neq}$, then (X, τ) is $T_2^{\neq}$ for any open base τ compatible with $-$.*

Proof. Suppose that (X, τ) is $T_2^{\neq}$ for some open space τ compatible with $-$, and let $x \neq y$. Then there exist $U, V \in \tau$ with $U \cap V = \emptyset$ such that $x \in U$ and $y \in V$. Since $U \subset \neg\neg U$ and $V \subset \neg\neg V$, we have $x \in -\neg U$, and $-\neg U \cap -\neg V = \emptyset$. Therefore (X, τ_{-}^{w}) is $T_2^{\neq}$.

Conversely, suppose that (X, τ_{-}^{w}) is $T_2^{\neq}$, that τ is an open base on X compatible with $-$, and that $x \neq y$. Then $x \in -\{y\}$ and $y \in -\{x\}$. In former case, because $x \in -\{y\} = -_{\tau}\{y\}$, there exists $U \in \tau$ such that $x \in U \subset \neg\{y\}$, and hence $x \in U \wedge \neg(y \in U)$. In the latter case, because $y \in -\{x\} = -_{\tau}\{x\}$, there exists $V \in \tau$ such that $y \in V \subset \neg\{x\}$, and hence $y \in V \wedge \neg(x \in V)$. Therefore $x \in U \wedge y \in V \wedge U \cap V = \emptyset$ for some $U, V \in \tau$. Thus (X, τ) is $T_2^{\neq}$ for any open base τ compatible with $-$. ■

Chapter 4

Conclusion

One could say that Bishop was well aware of the challenging nature of constructivising topology and he introduced the notion of working with neighbourhood spaces [5]. Not much development was immediately after him so it lied there dormant for a considerable number of years and it was only recently that Bridges and Vîţă [11–13] took up the challenge and it gave an impetus thereby reviving the interests amongst researchers which showed particularly in the works of Ishihara *et al.* [21, 25]. Of course the works of Sambin in formal topology [34], Aczel *et al.* [1, 2] on foundational matters, to mention a few, are of special interest in general.

The most noticeable fact in the constructivising process is the fewer 'tools' one is given to work with. To be specific, one has limited access to what a typical classical mathematicians would have. The examples in Chapter 1 showed the the operation $-_\tau$ could not satisfy conditions A1 and A5. Because of such limitations the notion of quasi–apartness was introduced to further the investigation into apartness spaces. As such, it was evident that an inequality would smoothen the development but Richman [32] interestingly showed how one could avoid working with inequality. Furthermore, we have witnessed that for each quasi–apartness space, one can construct the weakest and also the strongest neighbourhood structures that induce the given quasi–apartness spaces. It goes to show that the notion of a quasi–apartness space is more general than the notion of a (point–set) apartness space in which there is an adjunction between the category of neighbourhood spaces and the category of quasi-apartness spaces. Moreover, it is interesting to see that the constructions of limits and colimits in the category of neighbourhood spaces can be carried over to the category of quasi-apartness spaces under the adjunction [25].

One may be misled to believe that it is 'very hard' to push the development further but it is very clear that this is possible under a careful constructive formulation of the T_i separation properties T_i (for $i = 0, 1, 2$) with binary relations $\sim_i$ (or $\neq_i$) for a neighbourhood space X. For T_i^+ separation properties we can define in neighbourhood space of X such that $(\forall x, y \in X)[\neg(x \neq_i y) \implies x = y]$. Moreover, we already show that the separation properties can be carried over to induced and product spaces as in classical mathematics. It also shows that T_i^+ separation properties have some advantages over T_i separation properties on quasi–apartness spaces which can be dealt with in spaces with an inequality.

For future work, there are interesting questions one would raise after reading this short exposition.

- We have seen the development of T_i, for $i = 0, 1, 2$. What about for $i = 3$ and other higher order?

- How far one would be willing to push the boundary of this approach to other areas of topology? It isn't a matter of not being able to do it but rather a question of how difficult it is to do?

- Explore and see how feasible it is to study these concept under the context of proximity and nearness spaces? It would be interesting to see such development but highly likely the challenge there is far more greater when compared to working in an apartness space especially when sitting within a constructive framework.

Appendix A

Intuitionistic Logic

Working with a fixed first–order language $\mathcal{L}$, we adopt the primitive connectives $\vee$ (or),$\wedge$ (and),$\Rightarrow$ (implies), and $\neg$ (not). In presenting the following axioms, I assume familiarity with basic notions of elementary classical logic; details of these notions may be found in, for example,

Proposition Axiom

1. $\mathsf{p} \Rightarrow (\mathsf{p} \wedge \mathsf{p})$
2. $(\mathsf{p} \wedge \mathsf{q}) \Rightarrow (\mathsf{q} \wedge \mathsf{p})$
3. $(\mathsf{p} \Rightarrow \mathsf{q}) \Rightarrow (\mathsf{p} \wedge \mathsf{r} \Rightarrow \mathsf{q} \wedge \mathsf{r})$
4. $(\mathsf{p} \Rightarrow \mathsf{q}) \Rightarrow ((\mathsf{q} \Rightarrow \mathsf{r}) \Rightarrow (\mathsf{p} \Rightarrow \mathsf{r}))$
5. $\mathsf{q} \Rightarrow (\mathsf{p} \Rightarrow \mathsf{q})$
6. $(\mathsf{p} \wedge (\mathsf{p} \Rightarrow \mathsf{q})) \Rightarrow \mathsf{q}$
7. $\mathsf{p} \Rightarrow (\mathsf{p} \vee \mathsf{q})$
8. $(\mathsf{p} \vee \mathsf{q}) \Rightarrow (\mathsf{q} \vee \mathsf{p})$
9. $((\mathsf{p} \Rightarrow \mathsf{r}) \wedge (\mathsf{q} \Rightarrow \mathsf{r})) \Rightarrow (\mathsf{p} \vee \mathsf{q} \Rightarrow \mathsf{r})$

10. $\neg p \Rightarrow (p \Rightarrow q)$
11. $((p \Rightarrow q) \wedge (p \Rightarrow \neg q)) \Rightarrow \neg p$

The axioms of the **intuitionistic predicate calculus** are obtained by adding to the foregoing propositional axioms those in the following list, where $\forall$ and $\exists$ have their usual meanings. Note that $p[x/t]$ is the formula obtained on replacing every occurrence of x in p by t in accordance with standard conventions.

Predicate Axioms:

1. $\forall x(p \Rightarrow q) \Rightarrow (\forall x p \Rightarrow \forall x q)$
2. $\forall x(p \Rightarrow q) \Rightarrow (\exists x p \Rightarrow \exists x q)$
3. $p \Rightarrow \forall x p$ if x is not free in p
4. $\exists x p \Rightarrow p$ if x is not free in p
5. $\forall x p \Rightarrow p[x/t]$ if t is free for x in p
6. $p[x/t] \Rightarrow \exists x p$ if t is free for x in p
7. All generalisation of 1-6

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