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AXIOMATIC ANALYSIS VIA VALUE QUANTALES

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AXIOMATIC ANALYSIS VIA VALUE QUANTALES

by

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A thesis submitted in partial fulfillment of the requirements for the Degree of Master of Science in Mathematics

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July, 2014
Declaration of Authenticity

Statement by Author

I, Alveen Aditya Chand, declare that this thesis is my own work and that, to my best knowledge, it contains no materials previously published, or substantially overlapping with material submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

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The research in this thesis was performed under my supervision and to my knowledge is the sole work of Mr. Alveen Aditya Chand.

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Abstract

Since the introduction of the notion of metric space, mathematicians have shown immense interest towards it. Metric spaces represent an abstraction of the notion of distance, thus significant proportion of the research throughout the decades has been focused on its axioms. This thesis discusses a systematic analysis of quantale valued metric spaces (or $V$-spaces), a generalization of metric spaces which uses value quantales. A comparative study of basic results in Metric Analysis is conducted in regards to value quantales. Some significant results such as continuity of the binary operations on $V$-spaces are established along with results on sequential continuity, convergence of monotone nets and a generalized Squeeze Lemma. Some of these results enlighten on the importance of the properties of real numbers and assist in categorizing the properties as either general or special. Also, an example of the value quantale of distance distribution functions is presented in detail; such an example is hardly shown in existing literature. Furthermore, a construction of the completion of $V$-spaces which satisfy uniformly vanishing asymmetry (a weak form of symmetry) is given using Cauchy filters. This construction along with other results on $V$-spaces establish that quantale valued metric spaces give a powerful axiomatization of Analysis.
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Chapter 1

Introduction

1.1 Background

Analysis is one of the most fundamental pillars of modern mathematics. Its foundations were discovered by a French mathematician, Maurice Fréchet [16], who introduced the first abstract formulation of the notion of distance in his PhD dissertation dating 1906. This distance function was later on given the name 'metric' by Hausdorff in 1914 [48] and is defined below.

Definition 1.1.1. Let $X$ be a set. Then a distance function $d: X \times X \to [0, \infty]$ is a metric if it satisfies the following axioms:

1. Reflexivity: $d(x, x) = 0$.
2. Separated: if $d(x, y) = 0$, then $x = y$.
3. Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.
4. Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then the pair $(X, d)$ is a metric space, where $X$ is a set with $d$ being the metric on it.

Fréchet’s work marked the beginning of a very fruitful approach to Analysis unifying many seemingly disparate sporadic results under the single umbrella of Abstract Analysis. From then on, countless mathematicians have contributed to the advancement of this field and to generalizations of metric spaces. Some of these generalizations are obtained by relaxing certain axioms of metric spaces. For instance, if $d$ satisfies Axioms 1,2 and 4 from Definition 1.1.1, then $(X, d)$ is called a semi-metric space [6]. Similarly, by removing only Axiom 2, $(X, d)$ becomes a quasi-metric space. If both Axiom 2 and Axiom 3 are relaxed from Definition 1.1.1,
then \((X, d)\) is called a \textit{quasi-semi-metric space} \cite{15}. Many other variations of Definition 1.1.1 have been studied, even some stronger and weaker forms of the Triangle Inequality have been used. Throughout the decades, volumes of results have been compiled using these approaches and through this, we have attained a great deal of understanding on the importance of these underlying axioms.

In addition, in 1942, Menger introduced a probabilistic generalization of a metric space called a statistical metric space \cite{35}. Afterwards, through the contributions of Menger \cite{38, 37, 36}, Wald \cite{57}, Šerstnev \cite{50, 51}, and Schweizer, Sklar and Thorp \cite{49}, the useful notion of probabilistic metric space emerged and is defined in Example 3.1.6. The motivation behind this notion is that measuring distances in reality never yields an exact value, but always an approximation. Then in a probabilistic metric space, the distance between two points is a distance distribution function which gives a probability when values from \([0, \infty]\) are evaluated in the function. The probability refers to the closeness of the chosen value to the correct distance. A detailed literature on this is given in the book by Schweizer and Sklar \cite{48}

Afterwards, Lawvere showed in his ground breaking article \cite{33} in 1973, that generalized metric spaces are enriched categories, thus inspiring a lot of research on the reconciliation of Category Theory, metric spaces and Order Theory. Other generalizations of metric spaces which are well-known are uniform spaces and topological spaces. A uniform space is a topological space with additional (uniform) structure on it which permits sensible definitions of completeness and uniform convergence.

Complete metric and uniform spaces naturally inspire the question whether it is possible to obtain a complete space from an arbitrary metric or uniform space. Fortunately, standard constructions of completion are available for both arbitrary metric spaces \cite{6} and uniform spaces \cite{25} but things become more interesting when these spaces are generalized, i.e., certain axioms are either, relaxed, replaced or weakened. Over the decades, many mathematicians such as Howes \cite{24}, Render \cite{43}, Doitchinov \cite{10, 9}, Stoltenberg \cite{53}, Gregori, Mascarell, Sapena \cite{18}, Carlson, Hicks \cite{7}, Kivvu, Schellekens, Kunzi \cite{28, 29, 31}, Vickers \cite{56}, Bonsangue, Breugel, Rutten \cite{4}, Alemany, Salbany, Sanchez-Granero, Romaguera \cite{1, 45, 44}, Kopperman, Sunderhauf, Flagg \cite{12}, Schroeder \cite{47}, Lowen and Vaughan \cite{34} have introduced and published on the different types of constructions that arise in generalized metric and uniform spaces. In addition, a standard construction of completion also exists for probabilistic metric spaces given by Sherwood \cite{52}. 
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1.2 Motivation and Related Works

The approach to Abstract Analysis, which is nowadays used by most of the mathematics community, makes use of the real numbers in the sense that the real numbers represent the possible outcomes of a measurement of distance inherent in a metric space. As such, the abstraction is still limited and grounded in the familiar zone of real numbers. More precisely, the metric $d : X \times X \to [0, \infty]$ in a metric space uses values from the set of extended real numbers $[0, \infty]$ to represent distances between points in $X$. It is natural to eliminate the real numbers from the axiomatization which is precisely what Flagg did in introducing the notion of $V$-spaces where $V$ is a value quantale.

Research on this aspect of axiomatization can already be seen in the work of Kopperman [30] where he replaced $[0, \infty]$ with a value semi-group in the definition of metric space. Then Flagg, following in the footsteps of Kopperman, in the early 1990's, introduced continuity spaces [15, 14] which we will call quantale valued metric spaces (or simply $V$-space). This is to avoid confusion since Flagg and Kopperman both used the same term for their different axiomatizations. While the focus of Flagg’s research was Domain Theory in Theoretical Computer Science, the focus of our research is directed purely towards Abstract Analysis.

Mulvey [39] introduced the general notion of quantales in 1986. He used quantales in the context of Topology with the aim of providing a concrete non-commutative framework for the foundations of Quantum Mechanics. While the focus of this thesis is not in Theoretical Computer Science, it should be noted that $V$-spaces are of particular significance in the area of Domain Theory, a very important and active area of research related to Denotational Semantics. Applications are also found in probabilistic metric spaces which are examples of $V$-spaces. One final important application is in Ring Theory via the notion of structure spaces. This applications ties metric techniques to Abstract Algebra.

Much of the research on quantales till now has been focused towards Theoretical Computer Science, for use in Domain theory. Many mathematicians such as Rosenthal [46], Henriksen [20], Waszkiewicz, Hofmann [22, 23], Reis [21] and Bob Flagg [11] contributed to this research and much of their inspiration arose from the works of Kelly [26], Lawvere [33] and Kopperman [30, 13]. Then Flagg [15, 14] in particular, considered certain special quantales, called value quantales, that are designed to capture properties of $([0, \infty], \leq, +)$ which are very relevant in Analysis. His approach enables the use of intuition from ordinary metric spaces in the study of many other structures including topological spaces, probabilistic metric spaces,
CHAPTER 1. INTRODUCTION

and structure spaces. One of the objectives of this thesis is to construct precise relationships between $V$-spaces and existing notions of metric spaces and gain a broad understanding of the place of the theory of $V$-spaces in general mathematics.

Drawing on the rich source of ordinary Metric Analysis, of which $V$-spaces are a generalization, this thesis uses a suitable portion of the theory, and results are verified on the general setting. Every theorem proved for $V$-spaces immediately implies its validity for all the spaces which are examples of $V$-spaces. In addition, we are also interested in the axiom of symmetry in this thesis. Symmetry is crucial to the standard completion constructions and it has been the universal view for many decades that the constructions immediately break down when it is removed. We aim to present a completion construction of $V$-spaces and test the general view on symmetry for this case. The research for this construction is motivated from the standard completion construction of uniform spaces.

1.3 Results and Content

Research in mathematics can roughly be categorized in two types: problem solving and theory building. The direction of this thesis is the latter: to develop the theory of $V$-spaces. The second chapter introduces all the background concepts that are utilized in this thesis. This chapter is divided into three parts. The first section gives a detailed treatment of the basic notions of Category Theory, the language used to understand the results. Also included is a powerful and elegant result of Category Theory, the duality theorem, which simplifies many results in the thesis. The second part lists down the notions of Lattice Theory relevant to the structure of a value quantale. Given in order, we define poset, lattice, complete lattice, completely distributive lattice and value distributive lattice. In addition, we verify a few categorical results on lattices and complete lattices. Then, in the third section, we define quantales and give a comprehensive list of basic computational results. Then, we finally present the definition of a value quantale, a special type of quantale.

The third chapter mainly focuses on introducing $V$-spaces. This includes defining $V$-spaces and $V$-metrics and giving some important examples including the value quantale $V$ as a $V$-space and the standard product $V$-space. The latter part of chapter 3 consists of important results of uniform continuity for basic structure maps such as binary joins, meets and distance functions. Since sequences are not sufficient for $V$-spaces, we introduce nets and sieves in Chapter 4 and present basic results on the convergence of nets. In addition, we establish a general version of the Squeeze Lemma in Real Analysis. Chapter 5 is dedicated to the value quantale $Δ$
of distance distribution functions.

The penultimate chapter of the thesis, Chapter 6, presents a new completion construction of a $V$-space which satisfies a weak form of symmetry, called uniformly vanishing asymmetry (or UVA). We also discuss the role of symmetry and UVA in categorical terms for metric spaces and $V$-spaces. Chapter 7 consists of several interesting research problems that arise from the research we have done.

1.4 Short Summary of New Results

Chapter three onwards contains new results that do not appear in the existing literature on $V$-spaces. Below we briefly discuss the significance and implications of these results.

- Chapters three and four contain generalizations of the fundamentals of Real Analysis to the case of $V$-spaces. The main contribution of this body of results is in establishing that the value quantale axiomatization of Real Analysis is adequate and fruitful.

- Chapter five presents a detailed proof of one particular value quantale structure on the set $\Delta$ of distance distribution functions. In particular, we describe an addition operation given by convolution together with the proof of all the details in establishing it, thus endowing $\Delta$ with the structure of a value quantale. It appears that such a detailed presentation is not part of the current literature.

- Chapter six introduces a suitable notion of completeness of $V$-spaces and presents a new completion construction for $V$-spaces satisfying UVA.
Chapter 2

Preliminaries on Categories and Lattices

We introduce some concepts in this chapter that will assist in understanding the results of the thesis given from Chapter 3 onwards. This chapter is divided into three sections. The first section is dedicated to Category Theory, used throughout the thesis as the language of use. The next section contains basic Lattice Theory concepts such as poset, lattice, complete lattice and so on. These concepts give an overview of the underlying structure of value quantales which are thoroughly discussed in the last section of this chapter.

2.1 Category Theory

We begin by familiarizing ourselves with some Category Theory. Most of the material covered is very fundamental and can be easily found in graduate texts on the subject [32, 5]. For beginners, there are also some introductory notes on Category Theory [55, 54] which can be easily followed and understood.

**Definition 2.1.1.** A category $C$ is a collection of objects and a collection of morphisms (or arrows) $f: a \to b$, where $a$ and $b$ are objects, called the domain and co-domain of $f$, respectively. Further, for every object $a \in C$, there is an identity morphism $id_a: a \to a$ and for any pair of morphisms $f: a \to b$ and $g: b \to c$, there is a composite morphism $g \circ f: a \to c$ in $C$. In addition, the following properties are satisfied for all morphisms in $C$.

- For any morphism $f: a \to b$, it holds that

  $$ id_b \circ f = f = f \circ id_a; $$
• Given any three morphisms \( f: a \rightarrow b, g: b \rightarrow c \) and \( h: c \rightarrow d \) in \( C \), the associativity property

\[
    h \circ (g \circ f) = (h \circ g) \circ f
\]

holds.

**Remark 2.1.2.** We assume that there is a fixed set \( U \) called the *universe*. A category is said to be *small* if all its objects are elements of \( U \); otherwise \( C \) is said to be *large*.

**Example 2.1.3.** The category of sets, denoted \( \text{Set} \), consists of all sets as objects and all functions between sets as morphisms, which includes the identity function for each object and all the composites of the functions.

**Example 2.1.4.** The category of metric spaces, denoted \( \text{Met} \), consists of all metric spaces as objects and all uniformly continuous functions between metric spaces as morphisms, including the identity function for each object and all the composites of the functions. Similarly, we define the category \( \text{Unif} \) consisting of all uniform spaces as objects and all uniformly continuous functions between uniform spaces as morphisms.

**Example 2.1.5.** The category of pointed metric spaces, denoted \( \text{Met}^* \) consists of all pairs \( (X, x) \) as objects where \( X \) is a metric space and \( x \in X \) and all functions \( f: (X, x) \rightarrow (Y, y) \) such that \( f: X \rightarrow Y \) is uniformly continuous with the additional property that \( f(x) = y \). For each object \( (X, x) \), the identity function \( id_{(X,x)}: (X, x) \rightarrow (X, x) \) exists and for every pair of functions \( f: (X, x) \rightarrow (Y, y) \) and \( g: (Y, y) \rightarrow (Z, z) \), the composition \( g \circ f: (X, x) \rightarrow (Z, z) \) is also in \( \text{Met}^* \).

**Example 2.1.6.** The category of small categories, denoted by \( \text{Cat} \), consists of all small categories as objects and all functors between the small categories as morphisms. For every object \( C \), the identity functor \( id_C: C \rightarrow C \) exists in \( \text{Cat} \), and for every pair of functors \( F: A \rightarrow B \) and \( G: B \rightarrow C \), the composition \( G \circ F: A \rightarrow C \) is also in \( \text{Cat} \).

**Example 2.1.7.** Let \( B \) and \( C \) be categories. Then we define a category \( B \times C \) called the *product* of \( B \) and \( C \) with objects being the pairs \( (b, c) \) of objects \( b \in B \) and \( c \in C \). A morphism in \( B \times C \) is a pair \((f, g)\) where \( f: b \rightarrow b' \) is in \( B \) and \( g: c \rightarrow c' \).
is in \( C \). The identity morphism for an object \((b, c)\) is \((\text{id}_b, \text{id}_c)\) where \(\text{id}_b : b \to b\) and \(\text{id}_c : c \to c\), and the composition of morphisms in \( B \times C \) simply derives from the composition of morphisms in the categories \( B \) and \( C \).

Remark 2.1.8. The categories \( \text{Set}, \text{Met}, \text{Unif}, \text{Met}^* \) and \( \text{Cat} \) are large categories. If \( B \) and \( C \) are small categories, then product \( B \times C \) is small, otherwise large.

Definition 2.1.9. Let \( B \) and \( C \) be categories. Then we define a functor \( F : B \to C \) which consists of two suitably related operations: The object function \( F \) assigns to each object \( b \) an object \( Fb \in C \) given by \( b \mapsto Fb \) and the arrow function (also denoted by \( F \)) assigns to each arrow \( p : b \to b' \) in \( B \) to an arrow \( Fp : Fb \to Fb' \) in \( C \). These functions are required to preserve the identity and composition of arrows in \( B \), i.e.,

\[
F(\text{id}_b) = \text{id}_{Fb} \quad \text{and} \quad F(g \circ f) = Fg \circ Ff,
\]

where the latter holds whenever the composite \( g \circ f \) is defined in \( B \).

Example 2.1.10. Let \( B, C \) be categories and \( T : B \to C \) a functor. If \( T \) simply forgets some or all of the structure of the objects in the domain \( B \), then \( T \) is commonly called a forgetful functor. For instance, there is the functor \( T : \text{Met} \to \text{Set} \) which assigns to each metric space \( X \) the set \( TX \) of its elements (the metric \( d_X \) on the metric space is "forgotten") and assigns to each morphism \( f : X \to Y \) of metric spaces the same function \( f \) which is then regarded just as the function between sets.

Definition 2.1.11. Let \( B \) and \( C \) be categories and \( T : B \to C \) a functor. We say that the functor \( T \) is faithful when to every pair \( b, b' \) of objects of \( B \) and to every pair \( f_1, f_2 : b \to b' \) of parallel arrows of \( B \), the equality of the arrows \( Tf_1 = Tf_2 \) in \( C \) implies \( f_1 = f_2 \). The functor \( T \) is full when to every pair \( b, b' \) of objects of \( B \) and to every arrow \( g : Tb \to Tb' \) of \( C \), there is an arrow \( f : b \to b' \) of \( B \) with \( g = Tf \). When \( T \) is full and faithful, we say that it is fully faithful.

Example 2.1.12. Let \( C \) be a category. The diagonal functor \( \Lambda \) from \( C \) to the product category \( C \times C \) is defined on the objects and morphisms of \( C \) by \( \Lambda(c) = (c, c) \) and \( \Lambda(f) = (f, f) \), respectively. Clearly,

\[
\Lambda(\text{id}_c) = (\text{id}_c, \text{id}_c) = \text{id}_{(c, c)}
\]

and also the composition

\[
\Lambda(f \circ g) = (f \circ g, f \circ g) = (f, f) \circ (g, g) = \Lambda(f) \circ \Lambda(g),
\]

and so \( \Lambda \) is indeed a functor. It is easy to verify that the diagonal functor \( \Lambda \) is fully
faithful.

**Definition 2.1.13.** Given two functors $T, U : B \to C$, a *natural transformation* 

$$\tau : T \to U$$

is a function defined between the two functors which assigns to each object $b$ of $B$ an arrow $\tau_b : Tb \to Ub$ of $C$ such that for every arrow $f : b \to b'$ in $B$, the diagram below commutes.

\[
\begin{array}{ccc}
Tb & \xrightarrow{Tf} & Tb' \\
\downarrow{\tau_b} & & \downarrow{\tau_{b'}} \\
Ub & \xrightarrow{Uf} & Ub'
\end{array}
\]

**Example 2.1.14.** If the functors $U = T$, then we obtain the identity transformation of $T$, denoted $id_T : T \to T$, which describes the commutative diagram given below.

\[
\begin{array}{ccc}
Tb & \xrightarrow{Tf} & Tb' \\
\downarrow{id_b} & & \downarrow{id_{b'}} \\
Tb & \xrightarrow{Tf} & Tb'
\end{array}
\]

**Example 2.1.15.** Let $B$ and $C$ be categories. Then we can construct a *functor category*, denoted $C^B$, consisting of all functors $F : B \to C$ as objects and all the natural transformations between the functors as morphisms, which includes the identity transformation for each object and all the composites of the natural transformations in $C^B$.

**Definition 2.1.16.** Let $C$ be a category. Then an object $a$ is *initial* in $C$ if to each object $c$, there is exactly one arrow $a \to c$. An object $b$ is *terminal* in $C$ if to each object $c$ in $C$ there is exactly one arrow $c \to b$. If there is a element $d$ in $C$ such that $d$ is both initial and terminal in $C$, then $d$ is a *zero object* in $C$.

**Example 2.1.17.** For each set $S \in \text{Set}$, there is exactly one function from the empty set $\emptyset$ to $S$ which is precisely the empty function, thus $\emptyset$ is the initial object in $\text{Set}$. Any singleton set in $\text{Set}$ is easily verified to a terminal object. For the category $\text{Met}$, the initial object is the empty metric space $(X, d)$ and a terminal object is any metric space $(Y, d_Y)$, where $Y$ is a singleton set.

**Definition 2.1.18.** Let $C$ be a category. A *subcategory* $S$ of $C$ consists of a sub-collection of objects and a sub-collection of arrows of $C$ such that, with the induced identities and composition, all the properties of a category are satisfied. The functor
$T: S \rightarrow C$ is an inclusion functor which is an inclusion map that sends each object and each arrow of $S$ to itself in $C$. We say that $S$ is a full subcategory of $C$ when the inclusion functor $T$ is full.

**Example 2.1.19.** We define the categories $\text{SMet}$, $\text{QMet}$ and $\text{SQMet}$ spanned by semi-metric, quasi-metric and semi-quasi-metric spaces, respectively. The morphisms are uniformly continuous functions between the objects. Note that $\text{Met}$ is a full subcategory of $\text{SMet}$, $\text{QMet}$ and $\text{SQMet}$, and $\text{SMet}$ and $\text{QMet}$ are full subcategories of $\text{SQMet}$. Similarly, $\text{Unif}$ is a full subcategory of $\text{QUnif}$ which is spanned by quasi-uniform spaces.

**Definition 2.1.20.** Let $C$ be a category. Then the opposite category (or dual category) of $C$, denoted by $C^{\text{op}}$, consists of all objects of $C$ and the arrows $f^{\text{op}}: b \rightarrow a$ in $C^{\text{op}}$ for every arrow $f: a \rightarrow b$ in $C$. The composition $f^{\text{op}}g^{\text{op}} = (gf)^{\text{op}}$ is defined in $C^{\text{op}}$ for every pair of composable arrows $f$ and $g$.

**Definition 2.1.21.** Let $C$ be a category. Let $\Sigma$ be a statement about $C$ which is constructed out of terms of category theory, ordinary propositional connectives and the usual quantifiers. $\Sigma^\ast$, the dual statement of $\Sigma$, about $C^{\text{op}}$ is constructed by making replacements throughout $\Sigma$ such that all arrows in $\Sigma$ are reversed.

**Theorem 2.1.22.** [32] (The Duality Theorem) Let $C$ be a category and $\Sigma$ a statement about $C$. Then if $\Sigma$ is true about $C$, then the dual statement $\Sigma^\ast$ is true about $C^{\text{op}}$.

**Example 2.1.23.** The dual of an initial object in a category $C$ is a terminal object in $C^{\text{op}}$.

**Definition 2.1.24.** Let $C$ and $D$ be categories. If $S: D \rightarrow C$ is a functor and $c \in C$ an object, a universal arrow from $c \rightarrow S$ is a pair $(r, u)$ consisting of an object $r \in D$ and an arrow $u: c \rightarrow Sr$ of $C$, such that to every pair $(d, f)$, where $d$ is an object of $D$ and $f: c \rightarrow Sd$ an arrow of $C$, there is a unique arrow $g: r \rightarrow d$ of $D$ such that the following square

\[
\begin{array}{ccc}
c & \xrightarrow{u} & Sr \\
\downarrow{id} & & \downarrow{sg} \\
c & \xrightarrow{f} & Sd
\end{array}
\]

commutes. In simple terms, we say $u: c \rightarrow Sr$ is universal among morphisms $f: c \rightarrow Sd$ for objects $d \in D$. 

Definition 2.1.25. Given a category $C$, a small category $J$ called an index category, and the diagonal functor $\Lambda: C \to C^J$, a limit for the functor $F: J \to C$ in $C^J$ is a universal arrow $(r, v)$ from $\Lambda$ to $F$. It consists of an object $r$ of $C$, usually denoted

$$ r = \lim_{\leftarrow} F $$

and called the limiting object of the functor $F$, and a natural transformation $v: \Lambda r \to F$ which is universal among natural transformations $\tau: \Lambda c \to F$, for objects $c$ of $C$. This means for every $\tau: \Lambda c \to F$, there exists $g: c \to r$ such that $\tau = v \circ \Lambda g$. Diagrammatically:

\[
\begin{array}{ccc}
\Lambda c & \xrightarrow{\tau} & F \\
\Lambda g \downarrow & & \downarrow \text{id}_F \\
\Lambda r & \xrightarrow{v} & F
\end{array}
\]

If every functor $F: J \to C$ (where $J$ is small) has a limit, then $C$ is called a small complete category.

Example 2.1.26. If $J$ is a finite category, then a universal arrow from $\Lambda$ to $F$ is simply called a finite limit for functor $F$. If every functor $F$ has a finite limit, then $C$ is called a finite complete category.

Remark 2.1.27. A colimit is simply the dual of the notion of limit of a functor

$$ F: J \to C $$

and colimiting object is denoted by $\lim_{\rightarrow} F$. If there exists a colimit for every functor $F$, then $C$ is called a small cocomplete category. Similarly, finite colimit is the dual of finite limit and if every functor $F$ has a finite colimit, then $C$ is a finite cocomplete category.

Definition 2.1.28. Let $C$ be a category. A categorical product diagram of two objects $x$ and $y$ of $C$ consists of an object of $C$, called a product object and denoted as $x \amalg y$, together with two arrows

$$ x \xleftarrow{\pi_x} x \amalg y \xrightarrow{\pi_y} y $$

such that for every pair of arrows $f: z \to x$ and $g: z \to y$, there exists a unique arrow $h: z \to x \amalg y$ making the following diagram commute.

\[
\begin{array}{ccc}
x & \xleftarrow{\pi_x} & x \amalg y \\
\downarrow f & & \downarrow \pi_y \\
z & \xrightarrow{g} & y
\end{array}
\]
This is also equivalent to saying that $f = \pi_i \circ h$ and $g = \pi_j \circ h$. In addition, the object $x \amalg y$ is unique up to an isomorphism in $C$ and the arrows $\pi_i$ and $\pi_j$ are called projections of the product.

Remark 2.1.29. A categorical coproduct is simply the dual of the notion of categorical product given above. The dual of the projections are the arrows $\eta_x: x \to x \amalg y$ and $\eta_y: y \to x \amalg y$ called injections of the coproduct and the coproduct object is denoted by $x \amalg y$.

Remark 2.1.30. Note that the notions of categorical product and coproduct given above are binary, i.e., for any two objects in $C$. We can easily generalize these notions for a set $S$ of objects in $C$. In that case, for any $s \in S$, a projection of the generalized product is

$$\pi_s: \prod S \to s$$

and an injection for the generalized coproduct is

$$\eta_s: s \to \coprod S$$

where $\prod S$ and $\coprod S$ are the generalized product and coproduct objects, respectively.

Definition 2.1.31. Let $C$ be a category. Given two parallel arrows

$$\xymatrix{ x \ar@/^/[r]^f & y \ar@/^/[l]^g }$$

in $C$, then their equalizer $(w, k)$ consists of an object $w$ and an arrow $k: w \to x$ of $C$ such that

$$f \circ k = g \circ k.$$ 

Moreover, every arrow $h: v \to x$ such that $f \circ h = g \circ h$ factorizes uniquely through $w$. Diagrammatically:

$$\xymatrix{ v \ar[r]^h & x \ar@/_/[r]_g & y \ar@/^/[u]^h \ar[l]_f \ar[u]^k }$$

Note that the arrow $h': v \to w$ is unique in $C$ and the object $w$ is called the equalizing object.

Remark 2.1.32. The dual of an equalizer is called a coequalizer.
Definition 2.1.33. Let $C$ and $D$ be categories. Then $C$ and $D$ are isomorphic, denoted $C \cong D$, if there exists a pair of invertible functors $F : C \to D$ and $G : D \to C$ such that $F \circ G = I_D$ and $G \circ F = I_C$ where $I_C$ and $I_D$ are the identity functors of $C$ and $D$, respectively. In comparison, an equivalence between $C$ and $D$ is defined to be a pair of functors $F : C \to D, G : D \to C$ such that $I_C \cong G \circ F$ and $I_D \cong F \circ G$.

Definition 2.1.34. Let $C$ be a category and, $a$ and $b$ objects of $C$. Then the hom-set of $a$ and $b$, denoted $C(a, b)$, is the collection of all morphisms with domain $a$ and codomain $b$.

Definition 2.1.35. Let $C$ and $D$ be categories. Then an adjunction from $D$ to $C$ is a triple $(F, G, \varphi)$ consisting of two functors

\[
\begin{array}{c}
C \xrightarrow{F} D \\
\cong \\
\xleftarrow{G} \\
\end{array}
\]

and for every object $a \in C$ and every object $y \in D$, a bijection

\[\varphi_{a,y} : D(Fa, y) \cong C(a, Gy)\]

which is natural in the sense that, for every $h : a \to a'$ in $C$ and every $k : y \to y'$ in $D$, the diagrams

\[
\begin{array}{c}
D(Fa, y) \xrightarrow{\varphi} C(a, Gy) \\
\downarrow k \\
D(Fa, y') \xrightarrow{\varphi} C(a, Gy')
\end{array}
\]

and

\[
\begin{array}{c}
D(Fa, y) \xrightarrow{\varphi} C(a, Gy) \\
\downarrow Fh \\
D(Fa', y) \xrightarrow{\varphi} C(a', Gy)
\end{array}
\]

commutes. The functor $F$ is called the left adjoint for $G$ and $G$ is the right adjoint for $F$ or simply denoted by $F \dashv G$.

2.2 Lattice Theory

This section introduces basic Lattice theory notions and results which are also available in detail in [17, 3, 8].
**Definition 2.2.1.** A partial order on a set $A$ is a binary relation $\leq$ which satisfies the following properties. For every $x, y, z \in A$,

1. Reflexivity: $x \leq x$;
2. Antisymmetry: $x = y$ whenever $x \leq y$ and $y \leq x$;
3. Transitivity: if $x \leq y$ and $y \leq z$, then $x \leq z$.

The pair $(A, \leq)$ is called a partially ordered set or poset and is simply referred to as $A$ unless there may be any ambiguity, in which case the full notation will be used.

**Definition 2.2.2.** Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. Then a monotone function is a map $f: P \to Q$ which is order preserving, i.e., for every $x, y \in P$, $f(x) \leq_Q f(y)$ whenever $x \leq_P y$.

**Definition 2.2.3.** The category of posets, denoted by Pos, consists of all posets as objects and all monotone functions between posets as arrows. For a poset $P$, $id_P: P \to P$ is the identity function and composition is that of functions. Note that Pos is a large category.

**Remark 2.2.4.** It is easy to check that for each object $P$ in Pos, $id_P$ is monotone and that the composition of monotone functions is again monotone.

**Example 2.2.5.** A poset $P$ can be regarded as a special category on its own. In contrast to Pos, it is classified as a small category. The elements in the poset are the objects and the partial order on the poset represents the morphisms; there exists an arrow $a \to b$ if $a \leq b$ in $P$. The identity morphism $id_a: a \to a$ exists for all $a \in P$ since $P$ is reflexive and the composition of the morphisms also exist in $P$ by transitivity. Note that there can be at most one arrow between any two elements due to the partial order and the monotone functions between the posets are precisely functors. In fact, this correspondence extends to the equivalence of categories between Pos and the full subcategory of Cat spanned by the small categories with at most one arrow between any two objects.

**Definition 2.2.6.** Let $P$ be a poset. Then for every subset $A$ of $P$, a lower bound of $A$ is $l \in P$ such that for all $a \in A$, $l \leq a$ and an upper bound of $A$ is $u \in P$ such that for all $a \in A$, $a \leq u$.

In addition, a supremum (or least upper bound) of $A$, denoted by $\bigvee A$, is an upper bound of $A$ such that for all upper bounds $u$ of $A$, $\bigvee A \leq u$. An infimum (or
greatest lower bound) of $A$, denoted by $\bigwedge A$, is a lower bound of $A$ such that for all lower bounds $l$ of $A$, $\bigwedge A \geq l$.

We will use the short-hand notations, $a \lor b = \bigvee \{a, b\}$ and $a \land b = \bigwedge \{a, b\}$, which are called binary join and binary meet, respectively. Let $I$ be an indexing set and, $P(i)$ and $Q(i)$ be predicates for $i \in I$. Then the notation $\bigwedge_{Q(i)} P(i)$ is the same as $\bigwedge \{P(i) : a \text{ has the property } Q(i)\}$. Similarly, the notation $\bigvee_{Q(i)} P(i)$ simply means $\bigvee \{P(i) : a \text{ has the property } Q(i)\}$.

**Definition 2.2.7.** Let $P$ be a poset. Then a bottom element of $P$, denoted by $0_P$ (or simply $0$), is an element of the poset which satisfies $0_P \leq p$, for every $p \in P$ and a top element of $P$, denoted $\infty_P$ (or simply $\infty$), is an element of the poset which satisfies $p \leq \infty_P$ for every $p \in P$.

**Remark 2.2.8.** In Pos, the initial object is the empty poset and the terminal object is a poset containing a single element. The situation is different when a poset $P$ is viewed as a category, then $0$ is the initial object and $\infty$ is the terminal object.

**Proposition 2.2.9.** Let $P$ be a poset and $A \subseteq P$. Then $\bigvee A$ with the injections given in Example 2.2.5 is a categorical coproduct in $P$ viewed as a category.

**Proof.** The supremum of $A$, $\bigvee A$, by definition, satisfies for every $a \in A$, $a \leq \bigvee A$ which is the same as saying: $a \rightarrow \bigvee A$. Further, if $b$ is such that for every $a \in A$, $a \leq b$, i.e., $a \rightarrow b$, then $\bigvee A \leq b$, i.e., $\bigvee A \rightarrow b$, uniquely. \qed

**Definition 2.2.10.** A lattice $L = (L, \leq)$ is a poset in which for every finite subset $A \subseteq L$, $\bigvee A$ and $\bigwedge A$ exist in $L$. In categorical terms, a lattice is a poset which is finite complete and finite cocomplete.

**Remark 2.2.11.** It is a trivial fact that the bottom and top elements of a lattice are unique. Also, notice that for the null set $\emptyset$ in a lattice $L$, $\bigvee \emptyset$ and $\bigwedge \emptyset$ exist since the null set is finite, and $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = \infty$. So a lattice as defined above can not be empty and the smallest lattice is $\{0 = \infty\}$.

**Definition 2.2.12.** Let $L = (L, \leq_L)$ and $M = (M, \leq_M)$ be lattices. Then $f$ is a lattice homomorphism if the function $f : L \rightarrow M$ is monotone and preserves the bottom element, $f(0_L) = 0_M$.

**Remark 2.2.13.** It is easy to see that the monotonicity of the lattice homomorphism implies that for every $x, y \in L$, both

$$f(x) \lor_M f(y) \leq_M f(x \lor_L y)$$
and

\[ f(x \wedge_L y) \leq_M f(x) \wedge_M f(y) \]

hold.

Remark 2.2.14. The following definition of category of lattices deviates from the standard definition. This is since our principle aim is to study value quantales, thus we have defined the category of lattices to suit our study.

Definition 2.2.15. The category of lattices, denoted \( \text{Lat} \), consists of all lattices as objects and all lattice homomorphisms between lattices as arrows. Note that \( \text{Lat} \) is a large category.

Remark 2.2.16. It is easy to check that for each object \( L \) in \( \text{Lat} \), \( \text{id}_L \) is indeed a morphism and the composition of morphisms is again a morphism.

Remark 2.2.17. Given a lattice \( L \), we can easily verify that \( L^{\text{op}} \) is also a lattice. The element \( \infty \) in \( L \) is the bottom element in \( L^{\text{op}} \) and vice versa, and for any finite subset \( S \) in \( L \), the elements \( \vee S \) and \( \wedge S \) are simply \( \wedge S^{\text{op}} \) and \( \vee S^{\text{op}} \), respectively. So in regards to lattices, the duality theorem simply means that if any statement \( \Sigma \) is true for all \( L \), then \( \Sigma^{\text{op}} \) is also true for all \( L \).

Remark 2.2.18. Both the initial and the terminal objects of \( \text{Lat} \) is a lattice \( L \) where \( L = \{0 = \infty\} \) is a singleton set, thus \( L \) is a zero object of the category.

Proposition 2.2.19. Let \( L \) be a lattice. Assume that \( \wedge B, \vee B, \wedge C \) and \( \vee C \) exists for \( B, C \subseteq L \) and \( B \subseteq C \). Then the following are true:

1. For \( a \in L \), \( a \leq \wedge B \) if and only if for all \( b \in B \), \( a \leq b \).
2. For \( a \in L \), \( \vee B \leq a \) if and only if for all \( b \in B \), \( b \leq a \).
3. \( \wedge C \leq \wedge B \) and \( \vee B \leq \vee C \).
4. For any \( b \in B \), \( b \leq \vee B \) and \( \wedge B \leq b \).

Remark 2.2.20. The proofs are trivial and the properties are listed for easy reference.

Lemma 2.2.21. Let \( L \) be a lattice and \( T_i \) be a family of subsets of \( L \) indexed by \( I \). Then the following are true:
1. $\bigwedge \left( \bigcup_{i \in I} T_i \right) = \bigwedge_{j \in I} (\bigwedge T_j)$;

2. $\bigvee \left( \bigcup_{i \in I} T_i \right) = \bigvee_{j \in I} (\bigvee T_j)$.

**Proof.**

1. We will show that $\bigwedge \left( \bigcup_{i \in I} T_i \right) \leq \bigwedge_{j \in I} (\bigwedge T_j)$ and $\bigwedge \left( \bigcup_{i \in I} T_i \right) \geq \bigwedge_{j \in I} (\bigwedge T_j)$.

For the first inequality, through Proposition 2.2.19(1), it is enough to show that

$$\bigwedge \left( \bigcup_{i \in I} T_i \right) \leq \bigwedge T_j$$

for each $j \in I$. Since $T_j \subseteq \bigcup_{i \in I} T_i$ for each $j \in I$, by Proposition 2.2.19(3) the desired result follows. For the second inequality, through Proposition 2.2.19(1), it is enough to show that for any $t \in \bigcup_{i \in I} T_i$,

$$t \geq \bigwedge_{j \in I} (\bigwedge T_j).$$

Let $t \in \bigcup_{i \in I} T_i$. Then $t \in T_j$ for some $j \in I$, thus $t \geq \bigwedge T_j$. Since

$$\bigwedge T_j \in \{ \bigwedge T_i \mid i \in I \},$$

then by Proposition 2.2.19(3), we obtain

$$t \geq \bigwedge T_j \geq \bigwedge \left( \bigwedge_{i \in I} T_i \right).$$

2. We note that the property is the dual of part (1). Then by Remark 2.2.17, the property holds in $L$.

**Definition 2.2.22.** Let $L_1$ and $L_2$ be lattices and $f : L_1 \to L_2$ a monotone function. Then a **right adjoint** of $f$ is a monotone function $g : L_2 \to L_1$, such that for every $x \in L_1$ and $y \in L_2$,

$$f(x) \leq y \text{ if and only if } x \leq g(y).$$

Then $f$ is called the **left adjoint** of $g$ denoted $f \dashv g$.

**Remark 2.2.23.** The definitions of left and right adjoints for lattices are derived from Definition 2.1.35.

**Theorem 2.2.24.** Let $L_1$ and $L_2$ be lattices. Then the following statements are true:
1. If \( f : L_1 \to L_2 \) is order preserving and
\[ g_1, g_2 : L_2 \to L_1 \]
are both right adjoints of \( f \), then \( g_1 = g_2 \).

2. If \( g : L_2 \to L_1 \) is order preserving and
\[ f_1, f_2 : L_1 \to L_2 \]
are both left adjoints of \( g \), then \( f_1 = f_2 \).

Proof.

1. For every \( x \in L_1 \) and \( y \in L_2 \), we have
\[ f(x) \leq y \text{ if and only if } x \leq g_1(y) \]
and
\[ f(x) \leq y \text{ if and only if } x \leq g_2(y). \]
Thus for all \( y \in L_2 \),
\[ g_1(y) \leq g_1(y) \iff f(g_1(y)) \leq y \iff g_1(y) \leq g_2(y) \]
and
\[ g_2(y) \leq g_2(y) \iff f(g_2(y)) \leq y \iff g_2(y) \leq g_1(y). \]
The inequalities, \( g_1(y) \leq g_2(y) \) and \( g_2(y) \leq g_1(y) \) imply that \( g_1(y) = g_2(y) \) by antisymmetry.

2. We note the statement is the dual of (a). Then by Remark 2.2.17, the left adjoints \( f_1 \) and \( f_2 \) of \( g \) are equal.

\[ \Box \]

Definition 2.2.25. A complete lattice is a lattice \( L = (L, \leq) \) such that for any subset \( A \subseteq L \), \( \bigvee A \) and \( \bigwedge A \) exist in \( L \).

Remark 2.2.26. A complete lattice can also be referred to as small complete and small cocomplete when viewed as a category.
The following definition of the category of complete lattices deviates from the standard definition.

**Definition 2.2.27.** The category of complete lattices, denoted as \( \text{CLat} \), is the full subcategory of \( \text{Lat} \) spanned by the complete lattices.

**Remark 2.2.28.** The morphisms of \( \text{CLat} \) are just lattice homomorphisms, thus for every subset \( A \subseteq L \), it holds that \( \bigvee f(A) \leq f(\bigvee A) \) and \( f(\bigwedge A) \leq \bigwedge f(A) \) by the property of monotonicity of the morphism \( f \).

**Remark 2.2.29.** A complete lattice \( (L, \leq) \) where \( L = \{0 = \infty\} \) is a singleton set is both initial and terminal in \( \text{CLat} \), so \( (L, \leq) \) is a zero object in \( \text{CLat} \). Comparing with \( \text{Lat} \), we see that both have zero objects with the same underlying set.

**Lemma 2.2.30.** [17] Let \( P \) be a poset. If \( \bigwedge S \) exists for every \( S \subseteq P \), then \( P \) is a complete lattice.

**Remark 2.2.31.** With regards to the above result, note that the join \( \bigvee S \) can be defined in terms of meets, i.e.,

\[
\bigvee S = \bigwedge \{ p \in P \mid s \leq p, \forall s \in S \}.
\]

A complete proof is given in 'Lattice Theory' [17]. In addition, the result is very useful and significantly decreases the amount of work required to verify a complete lattice.

**Lemma 2.2.32.** Let \( L_1, L_2 \) be complete lattices in \( \text{CLat} \) and \( f : L_1 \to L_2 \) be monotone. Then \( f \) has a right adjoint if and only if for all \( S \subseteq L_1 \),

\[
f(\bigvee S) = \bigvee (f(S)).
\]

**Proof.** We firstly show that if \( f \) has a right adjoint, then \( f(\bigvee S) = \bigvee (f(S)) \).

Let \( g : L_2 \to L_1 \) be a right adjoint of \( f \). To show \( f(\bigvee S) = \bigvee (f(S)) \), we show that \( f(\bigvee S) \leq \bigvee (f(S)) \) and \( f(\bigwedge S) \geq \bigwedge (f(S)) \). For the first inequality, notice that \( f(\bigvee S) \leq \bigvee (f(S)) \) is equivalent to

\[
\bigvee S \leq g(\bigvee f(S)).
\]

Further, by Proposition 2.2.19(2), \( \bigvee S \leq g(\bigvee f(S)) \) if and only if \( s \leq g(\bigvee f(S)) \) for all \( s \in S \). But then \( s \leq g(\bigvee f(S)) \) is equivalent to \( f(s) \leq \bigvee (f(S)) \) which holds by Proposition 2.2.19(4). By Remark 2.2.28,

\[
f(\bigvee S) \leq f(\bigvee S)
\]
Next we show the converse of the above, i.e., if \( f(\bigvee S) = \bigvee f(S), \forall S \subseteq L \), then \( f \) has a right adjoint. We construct \( g : L_2 \rightarrow L_1 \) explicitly. Let \( y \in L_2 \), then define

\[
g(y) = \bigvee \{ x \in L_1 \mid f(x) \leq y \}.
\]

Now, we are left to show that \( g \) is monotone and is a right adjoint of \( f \). We first show that \( g \) is monotone. Let \( y_2 \leq y_1 \) in \( L_2 \). We have

\[
\{ x \in L_1 \mid f(x) \leq y_2 \} \subseteq \{ x \in L_1 \mid f(x) \leq y_1 \}.
\]

Then by Proposition 2.2.19(3), we obtain

\[
\bigvee \{ x \in L_1 \mid f(x) \leq y_2 \} \leq \bigvee \{ x \in L_1 \mid f(x) \leq y_1 \},
\]

or equivalently, \( g(y_2) \leq g(y_1) \) thus \( g \) is monotone. Next, we show that \( g \) is a right adjoint of \( f \). Assume that \( f(a) \leq b \), we wish to show that \( a \leq g(b) \). By definition of \( g(b) \) as a join, it follows from \( f(a) \leq b \) that \( a \leq g(b) \). Conversely, let \( a \leq g(b) \). Then, applying \( f \) and using the fact that \( f(\bigvee S) = \bigvee f(S) \) and \( f \) is monotone, we obtain

\[
\begin{align*}
f(a) & \leq f(g(b)) \\
& = f(\bigvee \{ x \in L_1 \mid f(x) \leq b \}) \\
& = \bigvee f(\{ x \in L_1 \mid f(x) \leq b \}) \\
& \leq b
\end{align*}
\]

and the proof is complete.

\[\square\]

**Lemma 2.2.33.** Let \( L_1, L_2 \) be complete lattices and \( f : L_1 \rightarrow L_2 \) be order preserving. Then \( f \) has a left adjoint if and only if for all \( S \subseteq L_1 \),

\[
f(\bigwedge S) = \bigwedge (f(S)).
\]

**Proof.** This is the dual statement of Lemma 2.2.32.

\[\square\]

**Definition 2.2.34.** Assume \( L \) is a poset and \( x, y \in L \). Then \( x \) is well above \( y \), denoted by \( x \succ y \), if for all subsets \( A \subseteq L \), if \( y \geq \bigwedge A \), then for some \( a \in A \), \( x \geq a \).

**Proposition 2.2.35.** Let \( L \) be a complete lattice. Then for all \( a, b, c \in L \),

1. If \( a \succ b \), then \( a \geq b \).
2. If \( a \succ b \) and \( b \geq c \), then \( a \succ c \).

3. If \( a \geq b \) and \( b \succ c \), then \( a \succ c \).

4. If \( a \succ b \) and \( b \succ c \), then \( a \succ c \).

Proof.

1. Let \( S = \{b\} \). Since \( \bigwedge S = b \), we have \( \bigwedge S = b \leq b \) and since \( a \succ b \), there is some \( s \in \{b\} \) such that \( s \leq a \). But \( s \in \{b\} \) means \( s = b \).

2. Assume that \( \bigwedge S \leq c \) for some set \( S \). But then \( \bigwedge S \leq b \), and since \( a \succ b \), it follows that \( s \leq a \) for some \( s \in S \). Thus if \( \bigwedge S \leq c \), then \( s \leq a \) for some \( s \in S \).

3. Let \( S \) be a set such that \( \bigwedge S \leq c \). Since \( b \succ c \), there is some \( s \in S \) such that \( b \geq s \). Since \( a \geq b \), then by transitivity, we obtain \( a \geq s \), therefore \( a \succ c \).

4. Let \( S \) be a set such that \( \bigwedge S \leq c \). Since \( b \succ c \), there is some \( s \in S \) such that \( b \geq s \). Since \( a \succ b \), then by part (1), we have \( a \geq b \), thus by transitivity, we obtain \( a \geq s \), therefore \( a \succ c \).

\[ \square \]

Definition 2.2.36. A completely distributive lattice \( L = (L, \leq) \) is a complete lattice which satisfies

\[
\bigwedge \bigvee_{i \in I} x_{i,j} = \bigvee_{f \in F} \bigwedge_{i \in I} x_{i,f(i)}
\]

for all doubly indexed families of elements \( \{x_{i,j} : i \in I, j \in J\} \), where \( F \) is the set of all choice functions for the family of sets \( \{J_i : i \in I\} \).

The following result is highly non-trivial and is utilized whenever the property of complete distributivity is mentioned in the latter chapters.

Theorem 2.2.37. \([40,~41,~42]\) Let \( L \) be a complete lattice. Then the following statements are equivalent.

1. \( L \) is completely distributive.

2. For all \( y \in L \), \( y = \bigwedge \{x \in L \mid x \succ y\} \).

Definition 2.2.38. A value distributive lattice \( (L, \leq) \) is a completely distributive lattice which satisfies that if \( p \succ 0 \) and \( q \succ 0 \), then \( p \land q \succ 0 \) for all \( p, q \in L \).

Remark 2.2.39. The definition of value distributive lattice is slightly different from the one introduced by Flagg [15]. He also includes a condition that the top element is well above the bottom element, i.e., \( \infty \succ 0 \) but this condition eliminates the trivial value distributive lattice \( \{0 = \infty\} \) which we see no reason to discard.
2.3 Value Quantales

This section is dedicated to defining value quantales, which are crucial to this thesis and also gives a few fundamental examples and results. In addition, a list of basic computational results are presented.

Definition 2.3.1. [14, 15] A quantale $V = (V, \leq, +)$ is a complete lattice with an associative and commutative operation $+: V \times V \rightarrow V$ satisfying the following.

1. $0$ is a unit for $+$, that is, for all $p \in V$,
   $$p + 0 = p;$$

2. (Infinite Distributive Law) For all $p \in V$ and all families $\{q_i\}_{i \in I}$ of elements of $V$,
   $$p + \bigwedge_{i \in I} q_i = \bigwedge_{j \in I} (p + q_j).$$

Definition 2.3.2. Let $V$ be a quantale and $p \in V$. Then for any map $p + \Box: V \rightarrow V$, there is a left adjoint $\Box - p: V \rightarrow V$ called subtraction which satisfies the universal property:

$$q \leq p + r \text{ if and only if } q - p \leq r.$$ 

Remark 2.3.3. The existence of the left adjoint is guaranteed by Lemma 2.2.33. The construction of the left adjoint shown in the proof of the Lemma can be used to give a explicit formula for this subtraction namely:

$$p - q = \bigwedge \{r \in V \mid p \leq q + r\}.$$ 

Example 2.3.4. The set of extended real numbers $[0, \infty]$ with the usual ordering $\leq$ is linearly ordered and completely distributive. Then defining the addition operation on the structure to be the usual addition on numbers, we obtain a quantale $([0, \infty], \leq, +)$. Then subtraction in this quantale is simply $p - q = \max\{0, p - q\}$ using the usual subtraction on numbers in $\mathbb{R}$.

Proposition 2.3.5. (Consequences of the Infinite Distributive Law) Let $V$ be a quantale. Then the following statement are true.

1. For $p \in V$ and $\infty$ the top element of $V$, $p + \infty = \infty$.

2. For all $p, q, r, s \in V$, if $p \leq q$ and $r \leq s$, then $p + r \leq q + s$. 

Proof.

1. Note that $\bigwedge \emptyset = \infty$ noted in Remark 2.2.11. Then by the Infinite Distributive Law given in Definition 2.3.1, we obtain
   \[ p + \infty = p + \bigwedge \emptyset = \bigwedge (p + \emptyset) = \bigwedge \emptyset = \infty. \]

2. Assume that $p \leq q$ and $r \leq s$. Then by Definition 2.3.2, $p + r \leq p + r$ is equivalent to $(p + r) - r \leq p$. Since $p \leq q$, we have $(p + r) - r \leq q$, or equivalently,
   \[ p + r \leq q + r. \]
   We have a similar argument for $q + r \leq q + s$. Then by transitivity, we obtain
   \[ p + r \leq q + s. \]

Lemma 2.3.6. Let $V$ be a quantale. Then for all $p, q, r \in V$,

1. If $p \leq q$, then $(p - r) \leq (q - r)$.
2. If $p \leq q$, then $(r - p) \geq (r - q)$.

Proof.

1. To prove that $(p - r) \leq (q - r)$, it is equivalent to show that $p \leq (q - r) + r$ by Definition 2.3.2. This follows from the property $q \leq (q - r) + r$ or equivalently, $(q - r) \leq (q - r)$ which holds in $V$.

2. To prove that $(r - q) \leq (r - p)$, it is equivalent to show that $r \leq (r - p) + q$ by Definition 2.3.2. This follows from the property $r \leq (r - p) + p$ by Proposition 2.3.5(2), which is equivalent to $(r - p) \leq (r - p)$ which holds in $V$.

Theorem 2.3.7. (Basic computations of quantales) Assume $V$ is a quantale. Then for all $p, q, r \in V$,

1. $q - p = 0$ iff $q \leq p$;
2. $q \leq p + (q - p)$ or equivalently, $q - (q - p) \leq p$;
3. $(p + q) - p \leq q$;
4. $((r - q) - p) = (r - (p + q)) = ((r - p) - q)$;
5. \((p + q) - (q + r) \leq (p - r)\);
6. \((p - r) + q \geq (p + q) - r\);
7. \(p + (q - r) \geq (p + q) - r\);
8. \((q - p) \leq (q - r) + (r - p)\);
9. \((p \wedge q) + (q \wedge r) = (p + q) \wedge (p + r) \wedge (q + q) \wedge (q + r)\);
10. \((p + q) - (q + r) \leq (p - q) + (q - r)\);
11. \((p - q) \leq p \leq (p + q)\).

**Proof.** Let \(p, q, r \in V\) where \(V\) is a quantale.

1. By Definition 2.3.2, we have the following equivalences.

\[
q - p \leq 0 \iff q \leq p + 0 \\
\iff q \leq p.
\]

2. By Definition 2.3.2, we have the following equivalences.

\[
q - p \leq q - p \iff q \leq p + (q - p) \\
\iff q - (q - p) \leq p.
\]

3. By Definition 2.3.2, we have the following equivalence.

\[
p + q \leq p + q \iff (p + q) - p \leq q.
\]

4. For the equality \(((r - q) - p) = (r - (p + q)) = ((r - p) - q)\), we will show that

(a) \(((r - q) - p) = (r - (p + q))\) and (b) \((r - (p + q)) \neq ((r - p) - q)\).

(a) For \(((r - p) - q) = (r - (p + q))\), it is enough to show that

\[
((r - p) - q) \leq (r - (p + q))
\]

and

\[
((r - p) - q) \geq (r - (p + q)).
\]

For the first inequality, by Definition 2.3.2, we obtain the following equivalences.

\[
((r - p) - q) \leq (r - (p + q)) \iff (r - p) \leq ((r - (p + q)) + q) \\
\iff r \leq (r - (p + q)) + (p + q)
\]
which holds by part (2) above. Also for the second inequality, by Definition 2.3.2, we obtain the following equivalences.

\[( (r-q) - p ) \geq r - (p+q) \iff ( (r-q) - p ) + (p+q) \geq r \]
\[\iff ( (r-q) - p ) + p \geq (r-q) \]
\[\iff ( (r-q) - p ) \geq ( (r-q) - p )\]

which holds due to the partial order.

(b) For the equality \((r - (p + q)) = ((r - p) - q)\), it is enough to show that

\[(r - (p + q)) \leq ((r - p) - q)\]

and

\[(r - (p + q)) \geq ((r - p) - q).\]

For the first inequality, by Definition 2.3.2, we obtain the following equivalences.

\[(r - (p + q)) \leq ((r - p) - q) \iff r \leq ((r - p) - q) + (p + q)\]
\[\iff (r - p) \leq ((r - p) - q) + q\]
\[\iff ((r - p) - q) \leq ((r - p) - q)\]

which holds due to the partial order. For the second inequality, by Definition 2.3.2, we obtain the following equivalences.

\[(r - (p + q)) \geq ((r - p) - q) \iff (r - (p + q)) + q \geq (r - p)\]
\[\iff (r - (p + q)) + (p + q) \geq r\]

which holds by the property given in part (2) above.

5. By the properties (3) and (4) above and Proposition 2.3.5, we obtain

\[(p + q) - (q + r) = ((p + q) - q) - r \leq p - r.\]

6. By the property (4) and (5) above, we obtain

\[(p + q) - (q + r) \leq (p - r) \iff ((p + q) - r) - q \leq (p - r)\]
\[\iff (p + q) - r \leq (p - r) + q.\]
7. Using property (6), we attain the following equivalences.

\[(q + p) - r \leq (q - r) + p \iff (q + p) - r \leq p + (q - r) \iff (p + q) - r \leq p + (q - r).\]

8. By the part (2) and (7) above, we obtain

\[q - p \leq ((q - r) + r) - p \leq (q - r) + (r - p).\]

9. By the distributive law of addition, we obtain

\[(p \land q) + (q \land r) = [(p + q) \land (p + r)] \land [(q + q) \land (q + r)] = (p + q) \land (p + r) \land (q + q) \land (q + r).\]

10. By joining the properties (8) and (5), we obtain

\[(p - q) + (q - r) \geq (p - r) \geq (p + q) - (q + r).\]

11. By Proposition 2.3.5(2) and Lemma 2.3.6, we obtain the inequalities \(p \leq p + q\) and \(p - q \leq p\). Thus by transitivity

\[(p - q) \leq p \leq (p + q).\]

\(\square\)

**Theorem 2.3.8.** Let \(V\) be a quantale. Then for \(p \in V\) and \(\{q_i\}_{i \in I}\) a family of elements of \(V\),

1. \(p - (\bigwedge_{i \in I} q_i) = \bigvee_{j \in I} (p - q_j)\);
2. \((\bigvee_{i \in I} q_i) - p = \bigvee_{j \in I} (q_j - p)\);

**Proof.**

1. We will show that

\[p - \left(\bigwedge_{j \in I} q_j\right) \geq \bigvee_{i \in I} (p - q_i)\]

and

\[p - \left(\bigwedge_{j \in I} q_j\right) \leq \bigvee_{i \in I} (p - q_i).\]
For the first inequality, it is enough to show that for each $i \in I$,

$$p - \left( \bigwedge_{j \in I} q_j \right) \geq (p - q_i)$$

which follows by $\bigwedge_{j \in I} q_j \leq q_i$ and Lemma 2.3.6(2).

For the second inequality $p - \left( \bigwedge_{j \in I} q_j \right) \leq \bigvee_{i \in I} (p - q_i)$, we will obtain the equivalence

$$p - \bigvee_{i \in I} (p - q_i) \leq \bigwedge_{j \in I} q_j.$$

Then by Proposition 2.2.19(1), the latter is equivalent to

$$p - \bigvee_{i \in I} (p - q_i) \leq q_j$$

for each $j \in I$ which, in turn, is equivalent to

$$(p - q_j) \leq \bigvee_{i \in I} (p - q_i)$$

which always holds.

2. We will show that

$$\left( \bigvee_{i \in I} q_i \right) - p \geq \bigvee_{j \in I} (q_j - p)$$

and

$$\left( \bigvee_{i \in I} q_i \right) - p \leq \bigvee_{j \in I} (q_j - p).$$

The first inequality is equivalent to $(\bigvee_{i \in I} q_i) - p \geq (q_j - p)$ for each $j \in I$ by Proposition 2.2.19(2), which easily follows from $q_j \leq \bigvee_{i \in I} q_i$ and Theorem 2.3.6(1).

Notice that the second inequality $(\bigvee_{i \in I} q_i) - p \leq \bigvee_{j \in I} (q_j - p)$ is equivalent to

$$\bigvee_{i \in I} q_i \leq \bigvee_{j \in I} (q_j - p) + p.$$

Then by Theorem 2.2.19(2), it is enough to show that for each $i \in I$,

$$q_i \leq \bigvee_{j \in I} (q_j - p) + p,$$

or equivalently, $q_i - p \leq \bigvee_{j \in I} (q_j - p)$ which always holds.
Definition 2.3.9. Let \( V = (V, \leq_V, +_V) \) and \( W = (W, \leq_W, +_W) \) be quantales. Then a quantale homomorphism is a function \( f: V \to W \) which is monotone, preserves the bottom element, i.e., \( f(0_V) = 0_W \) and satisfies \( f(a +_V b) \leq f(a) +_W f(b) \) for all \( a, b \in V \).

Definition 2.3.10. The category of quantales, denoted \( \textbf{Quant} \), consists of all quantales as objects and all quantale homomorphisms between the objects as morphisms. For a quantale \( V \), \( \text{id}_V: V \to V \) is the identity function and composition is that of functions.

Definition 2.3.11. A value quantale \( V = (V, \leq, +) \) is a quantale where the structure \( (V, \leq) \) is a value distributive lattice.

Example 2.3.12. For the completely distributive lattice \( ([0, \infty], \leq) \) given in Example 2.3.4, we notice that the well-above relation \( a > b \) defined in Definition 2.2.34 becomes simply \( a > b \) for \( a, b \in [0, \infty] \). Then \( ([0, \infty], \leq) \) is easily verified to be a value distributive lattice, thus \( ([0, \infty], \leq, +) \) is a value quantale where addition is the usual addition on numbers.

Example 2.3.13. On the value distributive lattice \( ([0, \infty], \leq) \) shown in example above, we endow a different addition operation given by

\[
    x \oplus y = \max\{x, y\}
\]

for any \( x, y \in [0, \infty] \), to obtain a value quantale \( ([0, \infty], \leq, \oplus) \). Note that the symbol \( \oplus \) is used instead of the usual \( + \) to differentiate the two value quantales defined on \([0, \infty]\).

Example 2.3.14. Let \( X \) be a set and \( \mathcal{P}_f(X) \) be the set of all finite subsets of \( X \). We define \( \omega \subseteq \mathcal{P}_f(X) \) to be a lower-set of \( X \) if for every \( A, B \in \mathcal{P}_f(X), A \in \omega \) and \( B \subseteq A \) implies that \( B \in \omega \). Then the set of all lower-sets of finite subsets of \( X \), denoted by \( \Omega(X) \), ordered by reverse inclusion is a complete lattice and in fact, a completely distributive lattice. With a well-above relation on \( \Omega(X) \) derived from Definition 2.2.34 and an addition operation defined by \( \omega_1 + \omega_2 = \omega_1 \cap \omega_2 \), we obtain a
value quantale \((\Omega(X), \leq, +)\) called free locales, denoted simply \(\Omega(X)\). More details on free locales is accessible in Flagg [15].

**Example 2.3.15.** The set \(\Delta\) of all the distance distribution functions together with the opposite of the point-wise ordering and the addition operation defined in Section 5.2 makes \((\Delta, \leq_{op}, +)\) a value quantale. Refer to Chapter 5 for a thorough treatment on \((\Delta, \leq_{op}, +)\).

**Definition 2.3.16.** The category of value quantales, denoted \(\textbf{VQuant}\), is a full subcategory of \(\textbf{Quant}\) spanned by the value quantales.

**Remark 2.3.17.** The morphisms of \(\textbf{VQuant}\) are simply the quantale homomorphisms. The property of monotonicity of the homomorphisms preserves the order of the well-above relation and also the meets and joins.

The result below presents an important property of value quantales.

**Theorem 2.3.18.** Let \(V\) be a value quantale. Then for every \(\epsilon > 0\) in \(V\) and any \(n \in \mathbb{Z}^+\), there exists a \(\delta > 0\) in \(V\) such that

\[ n \cdot \delta \leq \epsilon \]

where \(n \cdot \delta\) is the short notation of \(\delta\) added together \(n\) number of times.

**Proof.** Let \(\epsilon > 0\) and \(n \in \mathbb{Z}^+\). We define the sets

\[ S = \{n \cdot \delta \mid \delta > 0\} \]

and

\[ S' = \{\delta_1 + \ldots + \delta_n \mid \delta_1, \ldots, \delta_n > 0\}. \]

To show that \(\bigwedge S = 0\), we will show that (i) \(\bigwedge S' = 0\) and (ii) \(\bigwedge S' = \bigwedge S\).

(i) Fix \(\delta_1, \ldots, \delta_{n-1}\) so we have

\[ S_{\delta_1, \ldots, \delta_{n-1}} = \{\delta_1 + \ldots + \delta_n \mid \delta_n > 0\} = \delta_1 + \ldots + \delta_{n-1} + \{\delta_n \mid \delta_n > 0\}. \]

Since \(V\) is completely distributive, by Theorem 2.2.37, we have

\[ \bigwedge \{\delta_n \mid \delta_n > 0\} = 0, \]

then we obtain

\[ \bigwedge S_{\delta_1, \ldots, \delta_{n-1}} = \delta_1 + \ldots + \delta_{n-1} + \bigwedge \{\delta_n \mid \delta_n > 0\} = \delta_1 + \ldots + \delta_{n-1}. \]
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We can easily see that

\[ S' = \bigcup_{\delta_1, \ldots, \delta_{n-1} > 0} S_{\delta_1, \ldots, \delta_{n-1}}. \]

Then by Lemma 2.2.21 and Infinite Distributive Law, we have

\[
\bigwedge S' = \bigwedge \left( \bigcup_{\delta_1, \ldots, \delta_{n-1} > 0} S_{\delta_1, \ldots, \delta_{n-1}} \right) \\
= \bigwedge_{\delta_1, \ldots, \delta_{n-1} > 0} \left( \bigwedge S_{\delta_1, \ldots, \delta_{n-1}} \right) \\
= \bigwedge_{\delta_1, \ldots, \delta_{n-1} > 0} (\delta_1 + \ldots + \delta_{n-1}) \\
= \bigwedge \{\delta_1 | \delta_1 > 0\} + \ldots + \bigwedge \{\delta_{n-1} | \delta_{n-1} > 0\} = 0
\]

thus \( \bigwedge S' = 0 \).

(ii) We will next show that \( \bigwedge S = \bigwedge S' \). Since \( \bigwedge S' = 0 \) which is the bottom element of \( V \), we certainly have that \( \bigwedge S' \leq \bigwedge S \). We are left to show the other inequality, \( \bigwedge S \leq \bigwedge S' \), or equivalently, for all \( s' \in S' \), \( \bigwedge S \leq s' \) by Proposition 2.2.19(1).

Let \( s' \in S' \) and \( \delta_1, \ldots, \delta_n > 0 \) such that \( s' = \delta_1 + \ldots + \delta_n \). Since \( V \) is a value distributive lattice, we have \( \bigwedge_{k=1}^n \delta_k > 0 \). Denote \( \delta_m = \bigwedge_{k=1}^n \delta_k \). Then

\[
s' = \delta_1 + \ldots + \delta_n \geq n \cdot \delta_m \geq \bigwedge S.
\]

Since \( s' \in S' \) is arbitrary, it follows that \( \bigwedge S \leq \bigwedge S' \), thus

\[
\bigwedge S = \bigwedge \{n \cdot \delta | \delta > 0\} = \bigwedge S' = 0
\]

By Definition 2.2.34, there exists a \( \delta > 0 \) where \( n \cdot \delta \) is in \( S \) such that \( n \cdot \delta \leq \epsilon \).

For \( n = 2 \), we obtain the simplified result below, which is utilized extensively in the latter chapters.

**Lemma 2.3.19.** [15] Let \( V \) be a value quantale. Then for every \( \epsilon > 0 \) in \( V \), there exists a \( \delta > 0 \) in \( V \) such that

\[ \delta + \delta \leq \epsilon. \]
Chapter 3

Quantale Valued Metric Spaces

In this chapter, we introduce quantale valued metric spaces (or V-spaces), which are the main objects of study in this thesis. The first section simply defines V-spaces and provides some basic examples in Section 3.2. We establish a few results on the continuity of the basic structure functions, such as the distance functions on $X$ and $X^{op}$ and other binary operations. Note that when $V = [0, \infty]$, all the concepts and results for V-spaces specialize to the familiar notions of metric spaces.

3.1 General V-spaces

This section is dedicated to defining V-metrics and V-spaces. The notion of V-spaces generalizes metric spaces, ultrametric spaces, probabilistic metric spaces and structure spaces [15]. This generalization guarantees that every result obtained for V-spaces immediately implies its validity for the other spaces, thus we obtain a broad understanding of the structure of these spaces and V-spaces as well.

**Definition 3.1.1.** [14, 15] Let $V$ be a value quantale and $X$ a set. Then a function $d: X \times X \rightarrow V$ is called a V-metric if it satisfies the following properties. For every $x, y, z \in X$,

- Reflexivity: $d(x, x) = 0$ where 0 is the bottom element of $V$;
- Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

A *quantale valued metric space* (or V-space) is a pair $(X, d)$ where $X$ is a set and $d$ a V-metric, and is simply written by $X$ (but full notation is used if there is danger of confusion).
**Definition 3.1.2.** Let $X$ be a $V$-space. Then $X$ is *separated* if for all $x, y \in X$, $d(x, y) = 0$ and $d(y, x) = 0$ implies $x = y$. $X$ is *symmetric* if for all $x, y \in X$, $d(x, y) = d(y, x)$.

**Example 3.1.3.** Let $V$ be the value quantale $([0, \infty], \leq, +)$ given in Definition 2.3.12, then a $V$-space $X$ is a quasi-semi-metric space. If $X$ is also symmetric, then it is a semi-metric space. $X$ is precisely a metric space when $X$ is both symmetric and separated. For the definitions of quasi-semi-metric space, semi-metric and metric, refer to Section 1.1.

**Example 3.1.4.** If the value quantale $V$ is $([0, \infty], \leq, \oplus)$ given in Definition 2.3.13, then a $V$-space $X$ is a quasi-semi-ultrametric space. If $X$ is also symmetric, then it is a semi-ultrametric space and $X$ is exactly a ultrametric space when $X$ is both symmetric and separated.

**Example 3.1.5.** [15] Let $U$ be a set and $V$ be the value quantale $(\Omega(U), \leq, +)$ given in Definition 2.3.14, then a $V$-space $X$ is a quasi-semi-structure space. If $X$ is also symmetric, then it is a semi-structure space and $X$ is a structure space when $X$ is both symmetric and separated.

**Example 3.1.6.** [15] For the the value quantale $(\Delta, \leq_{op}, +)$ of distance distribution functions (defined in detail in Chapter 5), then a $V$-space $X$ is a quasi-semi-probabilistic metric space. If $X$ is also symmetric, then it is a semi-probabilistic metric space. $X$ is a probabilistic metric space when $X$ is both symmetric and separated.

**Proposition 3.1.7.** Let $(X, d)$ be a $V$-space. Then the functions $d^{op}$, $d^s$ and $d^{s+}$ defined by

1. $d^{op}(x, y) = d(y, x)$,
2. $d^s(x, y) = d(x, y) \lor d^{op}(x, y)$ and
3. $d^{s+}(x, y) = d(x, y) + d^{op}(x, y)$

are all $V$-metrics.

**Proof.** We will show that Reflexivity and the Triangle Inequality are satisfied for all three functions for every $x, y, z \in X$. 
1. For $d^{op}(x, y)$, we obtain the following.

(a) Reflexivity: $d^{op}(x, x) = d(x, x) = 0$.

(b) Triangle Inequality: We have

$$d^{op}(x, y) + d^{op}(y, z) = d(y, x) + d(z, y)$$

$$\geq d(z, x) = d^{op}(x, z).$$

2. Next, for $d^{s}(x, y)$, we obtain the following.

(a) Reflexivity: $d^{s}(x, x) = d(x, x) \lor d^{op}(x, x) = 0 \lor 0 = 0$.

(b) Triangle Inequality: We will show that

$$d^{s}(x, z) = [d(x, z) \lor d^{op}(x, z)]$$

$$\leq [d(x, y) \lor d^{op}(x, y)] + [d(y, z) \lor d^{op}(y, z)]$$

$$= d^{s}(x, y) + d^{s}(y, z)$$

Thus by Proposition 2.2.19(2), it is enough to show the following inequalities.

i. $d(x, z) \leq [d(x, y) \lor d^{op}(x, y)] + [d(y, z) \lor d^{op}(y, z)]$;

ii. $d^{op}(x, z) \leq [d(x, y) \lor d^{op}(x, y)] + [d(y, z) \lor d^{op}(y, z)]$.

For the first inequality, using Proposition 2.2.19(4) we obtain

$$d(x, z) \leq d(x, y) + d(y, z)$$

$$\leq [d(x, y) \lor d^{op}(x, y)] + [d(y, z) \lor d^{op}(y, z)].$$

Thus the second inequality is also true by interchanging the roles of the argument.

3. Now, for $d^{s+}(x, y)$, we obtain the following.

(a) Reflexivity: $d^{s+}(x, x) = d(x, x) + d^{op}(x, x) = 0 + 0 = 0$.

(b) Triangle Inequality: We see that

$$d^{s+}(x, z) = d(x, z) + d^{op}(x, z)$$

$$\leq [d(x, y) + d(y, z)] + [d^{op}(x, y) + d^{op}(y, z)]$$

$$= [d(x, y) + d^{op}(x, y)] + [d(y, z) + d^{op}(y, z)]$$

$$= d^{s+}(x, y) + d^{s+}(z, y).$$
Therefore we conclude that $d^{op}(x,y)$, $d^s(x,y)$ and $d^{s+}(x,y)$ are V-metrics.

**Definition 3.1.8.** Let $V$ be a value quantale and $X$ a V-space. Then

- The **dual space** of $X$ is the V-space $(X,d^{op})$, denoted by $X^{op}$.
- The **join symmetrization** of $X$ is the V-space $(X,d^s)$, denoted by $X^s$.
- The **additive symmetrization** of $X$ is the V-space $(X,d^{s+})$, denoted by $X^{s+}$.

**Definition 3.1.9.** Let $V$ be a value quantale. Assume $X = (X,d_X)$ and $Y = (Y,d_Y)$ are V-spaces and $f : X \to Y$ is a function. Then $f$ is

- **uniformly continuous** if for every $\epsilon > 0$ in $V$, there exists a $\delta > 0$ in $V$ such that for every $x_1, x_2 \in X$, $d(x_1, x_2) \leq \delta$ implies that
  \[ d(f(x_1), f(x_2)) \leq \epsilon. \]
- **non-expansive** if for every $x_1, x_2 \in X$,
  \[ d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2). \]

**Proposition 3.1.10.** $\text{Met}_V$ consisting of all V-spaces as objects and all uniformly continuous mapping between the objects as morphisms is a category.

**Proof.** We prove the claim in two parts.

1. For a V-space $(X,d_X)$, the identity function $id_X$ is in $\text{Met}_V$: The identity function $id_X : X \to X$ is defined by $id_X(x) = x$ for every $x \in X$. It is easily seen that $id_X$ is an isometry, i.e.,
  \[ d_X(id_X(x), id_X(x')) = d_X(x, x'), \]
  which implies that $id_X$ is uniformly continuous, thus $id_X \in \text{Met}_V$.

2. Given a pair of uniformly continuous functions $f : (X,d_X) \to (Y,d_Y)$ and $g : (Y,d_Y) \to (Z,d_Z)$ in $\text{Met}_V$, the composite $g \circ f$ is also in $\text{Met}_V$: It is well-known that composites of functions are also functions, so $g \circ f$ is a function, then we are only left to show that $g \circ f$ is uniformly continuous.
Let $\epsilon > 0$. Since $g$ is uniformly continuous, there exists a $\delta_1 > 0$ corresponding to $\epsilon$ and since $f$ is also uniformly continuous, there exists a $\delta_2 > 0$ corresponding to $\delta_1$. Suppose $d_X(x, x') \leq \delta_2$ for some $x, x' \in X$. Then $d_Y(f(x), f(x')) \leq \delta_1$ which implies that

$$d_Z(g \circ f(x), g \circ f(x')) = d_Z(g(f(x)), g(f(x'))) \leq \epsilon,$$

then $g \circ f$ is uniformly continuous, thus $g \circ f \in \text{Met}_V$.

\[\square\]

**Definition 3.1.11.** The category $\text{Met}_V^\Sigma$ is a full subcategory of $\text{Met}_V$, spanned by the symmetric $V$-spaces and all the corresponding uniformly continuous functions between these objects as arrows. Similarly, we define other full subcategories of $\text{Met}_V$, denoted $\text{SMet}_V$ and $\text{SMet}_V^\Sigma$, spanned by separated $V$-spaces, and separated symmetric $V$-spaces, respectively. Note that $\text{SMet}_V^\Sigma$ is a full subcategory of both $\text{Met}_V^\Sigma$ and $\text{SMet}_V$ and all these four categories are large.

**Remark 3.1.12.** Given $V = [0, \infty]$, we can easily notice that the category $\text{Met}_V$ is isomorphic to $\text{SQMet}$ while $\text{Met}_V^\Sigma$ is isomorphic to $\text{SMet}$.

**Remark 3.1.13.** The initial object of $\text{Met}_V$ is the empty $V$-space and the terminal object is any $V$-space $(X, d)$ where $X$ is a singleton set. For $\text{Met}_V^\Sigma$, $\text{SMet}_V$ and $\text{SMet}_V^\Sigma$, the initials objects are the empty symmetric $V$-space, empty separated $V$-space and empty separated symmetric $V$-space, respectively. The terminal objects for these categories are similar to $(X, d)$ defined above, with the difference that $d$ is a symmetric $V$-metric for $\text{Met}_V^\Sigma$, separated for $\text{SMet}_V$, and both symmetric and separated for $\text{SMet}_V^\Sigma$.

**Lemma 3.1.14.** Let $X$ be a $V$-space. If it holds that $d(x, y) \leq \epsilon$ for all $\epsilon > 0$ in $V$, then $d(x, y) = 0$.

**Proof.** We can easily see that by Proposition 2.2.19(1) and Theorem 2.2.37,

$$d(x, y) \leq \bigwedge \{\epsilon \in V \mid \epsilon > 0\} = 0.$$

\[\square\]

**Proposition 3.1.15.** Let $(X, d_X)$ and $(Y, d_Y)$ be $V$-spaces. Then the function on $X \times Y$ given by

$$d_{X \times Y}((a, r), (b, s)) = d_X(a, b) \lor d_Y(r, s)$$

for $a, b \in X$ and $r, s \in Y$ is a $V$-metric.
Proof. We will show that \( d_{X \times Y}((a, r), (b, s)) \) is reflexive and satisfies the Triangle Inequality.

1. Reflexivity: For every \( a \in X \) and \( r \in Y \),
\[
d_{X \times Y}((a, r), (a, r)) = d_X(a, a) \lor d_Y(r, r) = 0 \lor 0 = 0.
\]

2. Triangle Inequality: For every \( a, b, c \in X \) and \( r, s, t \in Y \), we will show that
\[
d_{X \times Y}((a, r), (c, t)) = d_X(a, c) \lor d_Y(r, t) \leq (d_X(a, b) \lor d_Y(r, s)) + (d_X(b, c) \lor d_Y(s, t))
\]
\[
= d_{X \times Y}((a, r), (b, s)) + d_{X \times Y}((b, s), (c, t)).
\]

By Proposition 2.2.19(2), it suffices to show the following inequalities.

(a) \( d_X(a, c) \leq (d_X(a, b) \lor d_Y(r, s)) + (d_X(b, c) \lor d_Y(s, t)) \) and
(b) \( d_Y(r, t) \leq (d_X(a, b) \lor d_Y(r, s)) + (d_X(b, c) \lor d_Y(s, t)) \).

For the first inequality, we obtain
\[
d_X(a, c) \leq d_X(a, b) + d_X(b, c)
\]
\[
\leq (d_X(a, b) \lor d_Y(r, s)) + (d_X(b, c) \lor d_Y(s, t))
\]
by Proposition 2.2.19(4). The second inequality follows similarly. Therefore
\[
d_{X \times Y}((a, b), (x, y))
\]
is a \( V \)-metric.

\( \square \)

**Definition 3.1.16.** Let \((X, d_X)\) and \((Y, d_Y)\) be \( V \)-spaces. Then \((X \times Y, d_{X \times Y})\) is called the standard product \( V \)-space where
\[
d_{X \times Y}((a, x), (b, y)) = d_X(a, b) \lor d_Y(x, y)
\]
is the standard product \( V \)-metric.

**Definition 3.1.17.** Let \((X \times Y, d_{X \times Y})\) be a \( V \)-space. Then the function \( \pi_X : X \times Y \to X \) defined by
\[
\pi_X(x, y) = x
\]
is called the projection to \( X \). The projection \( \pi_Y : X \times Y \to Y \) is similarly defined.
Theorem 3.1.18. Let \((X, d_X)\) and \((Y, d_Y)\) be \(V\)-spaces. Then \((X \times Y, d_{X \times Y})\) with the projections is a categorical product in \(\text{Met}_V\).

Proof. We begin by showing that the projections \(\pi_X\) and \(\pi_Y\) are morphisms of \(\text{Met}_V\), i.e., they are uniformly continuous maps. We see that for every \(x, x' \in X\) and \(y, y' \in Y\),

\[
d_X(\pi_X(x, y), \pi_X(x', y')) = d_X(x, x') \\
\leq d_X(x, x') \lor d_Y(y, y') \\
= d_{X \times Y}((x, y), (x', y')),
\]

so \(\pi_X\) is non-expanding thus uniformly continuous. The argument follows similarly for \(\pi_Y\). Then for every uniformly continuous map \(f: A \to X\) and \(g: A \to Y\) where \(A\) is a \(V\)-space, we show that there exists a unique uniformly continuous function \(h: A \to X \times Y\) such that \(\pi_X h = f\) and \(\pi_Y h = g\). So we define the map \(h\) by \(h(a) = (x_a, y_a)\) where \(a \in A\) and \((x_a, y_a) \in X \times Y\). Then applying the property \(\pi_X(h(a)) = f(a)\), we obtain

\[
\pi_X(h(a)) = \pi_X(x_a, y_a) = x_a = f(a).
\]

This shows that we are forced to take \(x_a\) as \(f(a)\). Similarly, applying the property \(\pi_Y(h(a)) = g(a)\), we obtain

\[
\pi_Y(h(a)) = \pi_Y(x_a, y_a) = y_a = g(a).
\]

This shows that we are forced to take \(y_a\) as \(g(a)\), thus the map \(h\) is uniquely defined as \(h(a) = (f(a), g(a))\). Next we show that \(h\) is a morphism in \(\text{Met}_V\), that is, \(h\) is uniformly continuous. Let \(\epsilon > 0\). Since \(f\) and \(g\) are uniformly continuous, there exist \(\delta_1 > 0\) for \(f\) and \(\delta_2 > 0\) for \(g\), both corresponding to \(\epsilon\). Note that \(\delta_1 \land \delta_2 > 0\) since \(V\) is a value distributive lattice, so set \(\delta = \delta_1 \land \delta_2\). Suppose that \(d_A(a, b) \leq \delta\).

This means \(d_A(a, b) \leq \delta \leq \delta_1\) implying

\[
d_X(f(a), f(b)) \leq \epsilon.
\]

We also have \(d_A(a, b) \leq \delta \leq \delta_2\), then

\[
d_Y(g(a), g(b)) \leq \epsilon.
\]
Then it follows that
\[
d_{X \times Y}(h(a), h(b)) = d_{X \times Y}((f(a), g(a)), (f(b), g(b))) = d_X(f(a), f(b)) \lor d_Y(g(a), g(b)) \leq \epsilon \lor \epsilon = \epsilon,
\]
so \(h\) is uniformly continuous, thus \(h\) is in \(\text{Met}_V\).

**Proposition 3.1.19.** Let \(V\) be a value quantale. Then the function \(d_{st}: V \times V \to V\) given by \(d_{st}(x, y) = x - y\) is a \(V\)-metric.

**Proof.** We will show that \(d_{st}(x, y)\) satisfies the properties of \(V\)-metric.

Reflexivity: For every \(v \in V\), \(d(v, v) = v - v = 0\).

Triangle Inequality: By Theorem 2.3.7(8), we obtain
\[
d(x, z) + d(z, y) = (x - z) + (z - y) \geq x - y = d(x, y).
\]

**Definition 3.1.20.** Let \(V\) be a value quantale and \(d_{st}(x, y) = x - y\) as above. Then
\[
V_{st} = (V, d_{st})
\]
is the standard \(V\)-space structure on \(V\).

### 3.2 Uniform Continuity of the Basic Structure Maps

We prove basic continuity results, in particular, generalizing the result that the metric \(d: X \times X \to [0, \infty]\) is uniformly continuous.

**Definition 3.2.1.** Let \((X, d_X)\) and \((Y, d_Y)\) be \(V\)-spaces. Then for a function \(f: X \to Y\) is continuous at a point \(c \in X\) if for any \(\epsilon \succ 0\) in \(V\), there exists a \(\delta \succ 0\) in \(V\) such that for all \(x \in X\), the inequality \(d_X(x, c) \leq \delta\) implies that \(d_Y(f(x), f(c)) \leq \epsilon\).

**Theorem 3.2.2.** Let \(V\) be a value quantale and \(X = (X, d)\) a \(V\)-space. Then the function \(d: X \times X^{op} \to V_{st}\) is uniformly continuous.

**Proof.** Let \(\epsilon \succ 0\). Then by Lemma 2.3.19, there exists \(\delta \succ 0\) such that \(\delta + \delta \leq \epsilon\). Assume that \(d_X \times X^{op}((a, x), (b, y)) \leq \delta\) for \(a, b, x, y \in X\), then we need to show that
\[
d_{st}(d(a, x), d(b, y)) \leq \epsilon.
\]
By Definition 3.1.16, we have

\[ d_{X \times X^{\text{op}}}(a, x), (b, y) = d(a, b) \lor d^{\text{op}}(x, y) \leq \delta \]

then \( d(a, b) \leq \delta \) and \( d^{\text{op}}(x, y) = d(y, x) \leq \delta \) by Proposition 2.2.19(2). We know that

\[ d_{st}(d(a, x), d(b, y)) = d(a, x) - d(b, y). \]

By the Triangle Inequality, we have \( d_{st}(d(a, x), d(b, y)) \leq d(a, x) - d(b, y) \leq \delta \) which is equivalent to

\[ d(a, x) - d(b, y) \leq d(a, b) + d(y, x) \leq \delta + \delta \leq \epsilon, \]

therefore \( d \) is uniformly continuous. \( \square \)

**Corollary 3.2.3.** Let \( X = (X, d) \) be a \( V \)-space. Then the function \( d^{\text{op}}: X^{\text{op}} \times X \to V^{\text{op}}_{st} \) is uniformly continuous.

*Proof.* This is the dual statement of Theorem 3.2.2. \( \square \)

**Corollary 3.2.4.** Let \( V \) be a value quantale and \( X = (X, d) \) a \( V \)-space. If \( X = X^{\text{op}} \), then the functions \( d : X \times X \to V_{st} \) and \( d^{\text{op}} : X \times X \to V^{\text{op}}_{st} \) are uniformly continuous.

*Proof.* Let \( X = X^{\text{op}} \). We see that the function \( d : X \times X \to V_{st} \) is equivalent to \( d : X \times X^{\text{op}} \to V_{st} \) which by Theorem 3.2.2, is uniformly continuous, thus

\[ d : X \times X \to V_{st} \]

is uniformly continuous. Similarly, the function \( d^{\text{op}} : X \times X \to V^{\text{op}}_{st} \) is equivalent to \( d^{\text{op}} : X^{\text{op}} \times X \to V^{\text{op}}_{st} \) which by Corollary 3.2.3 is uniformly continuous, thus

\[ d^{\text{op}} : X \times X \to V^{\text{op}}_{st} \]

is uniformly continuous. \( \square \)

**Corollary 3.2.5.** Let \( X = (X, d) \) be a \( V \)-space. If \( X = X^{\text{op}} \), then the functions \( d^{s} : X \times X \to V^{s}_{st} \) and \( d^{s+} : X \times X \to V^{s+}_{st} \) are uniformly continuous.

*Proof.* Let \( X = X^{\text{op}} \). It is easily verified that the join of uniformly continuous functions is also uniformly continuous, thus \( d^{s} \) is uniformly continuous since

\[ d^{s}(x, y) = d(x, y) \lor d^{\text{op}}(x, y) \]

where \( d \) and \( d^{\text{op}} \) are uniformly continuous by Corollary 3.2.4.

For \( d^{s+} \), we use the fact that the addition of uniformly continuous functions is also
uniformly continuous, thus \(d^{x+}\) is uniformly continuous since

\[
d^{x+}(x,y) = d(x,y) + d^{op}(x,y)
\]

where \(d\) and \(d^{op}\) are uniformly continuous by Corollary 3.2.4.

Theorem 3.2.6. Let \(V\) be a quantale. Then the binary operation meet

\[-\wedge- : V_{st} \times V_{st} \to V_{st}\]

is non-expansive, and therefore uniformly continuous.

Proof. By Lemma 2.3.6(1) and Theorem 2.3.8(1), we obtain

\[
d_{st}(x \wedge y, x' \wedge y') = (x \wedge y) - (x' \wedge y')
\]

\[
= ( (x \wedge y) - x') \vee ((x \wedge y) - y')
\]

\[
\leq (x - x') \vee (y - y')
\]

\[
= d_{st}(x, x') \vee d_{st}(y, y')
\]

\[
= d_{V_{st} \times V_{st}}((x, y), (x', y')).
\]

Corollary 3.2.7. Let \(V\) be a quantale. Then the binary operation join

\[-\lor- : V_{st}^{op} \times V_{st}^{op} \to V_{st}^{op}\]

is non-expanding and therefore uniformly continuous.

Proof. This is the dual statement of Theorem 3.2.6.

Theorem 3.2.8. Let \(V\) be a quantale. Then the binary operation

\[-\lor- : V_{st} \times V_{st} \to V_{st}\]

is non-expansive and therefore uniformly continuous.
Proof. By Theorem 2.3.8(2), we obtain

\[ d_{st}(x \lor y, x' \lor y') = (x \lor y) - (x' \lor y') \]
\[ = [x - (x' \lor y')] \lor [y - (x' \lor y')] \]
\[ \leq (x - x') \lor (y - y') \]
\[ = d_{V \times V}(x, y, (x', y')). \]

Corollary 3.2.9. Let \( V \) be a quantale. Then the binary operation join

\[- \lor - : V_{st}^s \times V_{st}^s \to V_{st}^s\]

is non-expansive and therefore uniformly continuous.

Proof. By Theorem 3.2.8 and Corollary 3.2.7, we obtain

\[ d_{st}(x \lor y, x' \lor y') = d_{st}(x \lor y, x' \lor y') \lor d_{st}^{op}(x \lor y, x' \lor y') \]
\[ = ((x \lor y) - (x' \lor y')) \lor ((x' - y') - (x \lor y)) \]
\[ \leq d_{V \times V}(x, y, (x', y')) \lor d_{V_{st} \times V_{st}}^{op}(x, y, (x', y')) \]
\[ = d_{st}(x, y, (x', y')). \]

Theorem 3.2.10. Let \( V \) be a value quantale. Then the binary operation

\[- + - : V_{st} \times V_{st} \to V_{st}\]

is uniformly continuous.

Proof. For any \( \epsilon > 0 \), by Lemma 2.3.19, there exists \( \delta > 0 \) such that \( \epsilon \geq \delta + \delta \). We need to show that for all \( (a, b), (x, y) \in V_{st} \times V_{st} \), if \( d_{V_{st} \times V_{st}}((a, b), (x, y)) \leq \delta \), then

\[ d_{st}(a + b, x + y) \leq \epsilon. \]

We know that

\[ d_{V_{st} \times V_{st}}((a, b), (x, y)) = (a - x) \lor (b - y) \leq \delta \]

and

\[ d_{st}(a + b, x + y) = (a + b) - (x + y). \]
Then by Theorem 2.3.7(4),(6) and (7), we obtain

\[ d_{st}(a + b, x + y) = (a + b) - (x + y) \]
\[ = ((a + b) - x) - y \]
\[ \leq ((a - x) + b) - y \]
\[ \leq (a - x) + (b - y) \]
\[ \leq [(a - x) \lor (b - y)] + [(a - x) \lor (b - y)] \]
\[ = d_{V_{st} \times V_{st}}((a, b), (x, y)) + d_{V_{st} \times V_{st}}((a, b), (x, y)) \]
\[ \leq \delta + \delta \leq \epsilon \]

as needed.
Chapter 4

Convergence in $V$-spaces

In this chapter, we define nets in a $V$-space, generalize results from Metric Analysis and discuss any discrepancies. The first section introduces nets and sieves, and also, show some basic results on these concepts. The second section presents results on convergence of addition, subtraction, meet and join of convergent nets. Then the last two sections discuss the continuity of functions and convergence in monotone nets in a $V$-space.

4.1 Convergence of Nets

We begin by defining nets and sieves and giving a few examples, followed by generalizations of standard results of convergence in metric spaces.

**Definition 4.1.1.** A directed set is a pair $(N, \leq)$ consisting of a set $N$ and a binary relation $\leq$ which is reflexive, transitive and upwards directed, i.e., for every $a, b \in N$, there exists $c \in N$ such that $a \leq c$ and $b \leq c$.

**Example 4.1.2.** For the value quantale $[0, \infty]$, the set $\mathbb{N}$ of all natural numbers is easily verified to be a directed set.

**Example 4.1.3.** Let $V$ be a value quantale. Then the subset $S \subseteq V$ consisting of all elements $\epsilon$ in $V$ such that $\epsilon \succ 0$ with the opposite ordering of $V$ is a directed set, because $V$ is a value distributive lattice.

**Definition 4.1.4.** Let $V$ be a quantale. Then a sieve is a subset $\sigma \subseteq V$ which is a directed set with the opposite ordering of $V$, and for every $\epsilon \succ 0$ in $V$, there exists an element $s \in \sigma$ such that $\epsilon \geq s$.

**Proposition 4.1.5.** If $V$ is a value quantale, then $V$ has at least one sieve.
Proof. For value quantale \( V \), the subset \( S \subseteq V \) defined in Example 4.1.3 is a sieve, i.e., for every \( \epsilon > 0 \) in \( V \), we have \( \epsilon \in S \) and \( \epsilon \leq \epsilon \).

\[ \square \]

**Definition 4.1.6.** If \( N \) is a directed set and \( X \) is any set, a net \( \{x_n\}_{n \in N} \) indexed by \( N \) with values in \( X \), assigns to every \( n \in N \) an element \( x_n \) in \( X \).

**Example 4.1.7.** Since the set of natural numbers \( \mathbb{N} \) is a directed set, then a sequence \( \{x_n\}_{n \in \mathbb{N}} \) of elements in \( \mathbb{R} \) is a net.

**Example 4.1.8.** Let \( V \) be a value quantale. Since a sieve \( \sigma \subseteq V \) is a directed set, then a set of elements \( \{x_v\}_{v \in \sigma} \) indexed by \( \sigma \) is a net.

**Definition 4.1.9.** Let \( V \) be a value quantale and \( X \) a \( V \)-space. Then a net \( \{x_n\}_{n \in N} \) in \( X \) is said to converge to \( x \in X \) (or simply \( x_n \to x \)) if for any \( \epsilon > 0 \), there exists a \( K \in N \) such that, for every \( n \geq K \), \( d(x_n, x) \leq \epsilon \). The element \( x \) is a limit of the net and is written as

\[ \lim_{n \in N} x_n = x. \]

**Remark 4.1.10.** Since \( X \) does not satisfy symmetry and separability, there are two consequences that we immediately notice. Firstly, a convergent net in \( X \) may not necessarily be convergent in \( X^{op} \). Secondly, a convergent net in \( X \) may have many limits (uniqueness of a limit is not guaranteed).

**Theorem 4.1.11.** Let \( V \) be a value quantale and \( X \) a \( V \)-space. If \( \{x_n\}_{n \in N} \) is a net in \( X \) and \( x_n \to x_0 \) in \( X \) and \( x_n \to x'_0 \) in \( X^{op} \), then \( d(x'_0, x_0) = 0 \).

**Proof.** Let \( \epsilon > 0 \). By Lemma 2.3.19, there is \( \delta > 0 \) such that \( \delta + \delta \leq \epsilon \). Then there exists \( K_1 \in N \) such that for every \( n \geq K_1 \),

\[ d(x_n, x_0) \leq \delta \]

and there exists \( K_2 \in N \) such that for every \( n \geq K_2 \),

\[ d^{op}(x_n, x'_0) \leq \delta. \]

Since \( N \) is a directed set, there is an element \( K \in N \) such that \( K_1 \leq K \) and \( K_2 \leq K \). Then since \( d^{op}(x_n, x'_0) = d(x'_0, x_n) \), we obtain for every \( n \geq K \),

\[ d(x'_0, x_0) \leq d(x'_0, x_n) + d(x_n, x_0) = d^{op}(x_n, x'_0) + d(x_n, x_0) \leq \delta + \delta \leq \epsilon. \]
Since \( \varepsilon > 0 \) is arbitrary, we conclude by Lemma 3.1.14 that \( d(x'_0, x_0) = 0 \).

\[ \square \]

Remark 4.1.12. If \( X \) is symmetric, i.e., \( X = X^{op} \), then for every net \( \{x_n\}_{n \in N} \) if \( x_n \to t \) and \( x_n \to u \), then \( d(t, u) = d(u, t) = 0 \). But since the property of separation (given by Definition 3.1.2) is absent, \( t \) and \( u \) can still be distinct points in \( X \).

The next result shows the implication of both symmetry and separation:

**Corollary 4.1.13.** Let \( V \) be a value quantale. If \( V \)-space \( X \) is symmetric and separated, then a net \( \{x_n\}_{n \in N} \) has at most one limit.

**Remark 4.1.14.** If \( V = [0, \infty] \), then Corollary 4.1.13 reduces to a generalization of the well-known uniqueness of limit theorem of convergent sequences in metric spaces \([6]\).

### 4.2 Limit Theorems and Continuity of Functions

In this section, we basically show results on the convergence of nets joined by different operations in \( V_{st} \) and then afterwards, give a generalized result of Squeeze Lemma for sequences in Analysis. Lastly, we provide a result on continuity of functions in \( V \)-spaces.

**Theorem 4.2.1.** Let \( V \) be a value quantale. If the net \( \{x_n\}_{n \in N} \) converges to \( x \) and the net \( \{y_n\}_{n \in N} \) converges to \( y \) in \( V_{st} \), then

1. the net \( \{x_n + y_n\}_{n \in N} \) converges to \( x + y \);
2. the net \( \{x_n \lor y_n\}_{n \in N} \) converges to \( x \lor y \);
3. the net \( \{x_n \land y_n\}_{n \in N} \) converges to \( x \land y \).

**Proof.**

1. Let \( \varepsilon > 0 \). By Lemma 2.3.19, there exists a \( \delta > 0 \) such that \( \delta + \delta \leq \varepsilon \). Then there exists \( K_1 \in N \) such that for all \( n \geq K_1 \),

\[
   d_{st}(x_n, x) = x_n - x \leq \delta
\]

and there exists \( K_2 \in N \) such that for all \( n \geq K_2 \),

\[
   d_{st}(y_n, y) = y_n - y \leq \delta.
\]
Since \( N \) is a directed set, there is an element \( K \in N \) such that \( K_1 \leq K \) and \( K_2 \leq K \). Then for all \( n \geq K \), we have

\[
d_{st}(x_n + y_n, x + y) = (x_n + y_n) - (x + y) = ((x_n + y_n) - x) - y \\
\leq ((x - x) + y_n) - y \\
\leq (x_n - x) + (y_n - y) = d_{st}(x_n, x) + d_{st}(y_n, y) \\
\leq \delta + \delta = \epsilon
\]

by Theorem 2.3.7(4), (6) and (7). Since \( \epsilon > 0 \) is arbitrary, the result follows.

2. Let \( \epsilon > 0 \). Since \( \{x_n\}_{n \in N} \) and \( \{y_n\}_{n \in N} \) are convergent, there exists \( K_1 \in N \) such that for all \( n \geq K_1 \),

\[
d_{st}(x_n, x) = x_n - x \leq \epsilon
\]

and there exists \( K_2 \in N \) such that for all \( n \geq K_2 \),

\[
d_{st}(y_n, y) = y_n - y \leq \epsilon.
\]

Since \( N \) is a directed set, there is an element \( K \in N \) such that \( K_1 \leq K \) and \( K_2 \leq K \). Then for all \( n \geq K \), we have

\[
d_{st}(x_n \lor y_n, x \lor y) = (x_n \lor y_n) - (x \lor y) = (x_n - (x \lor y)) \lor (y_n - (x \lor y)) \\
\leq (x_n - x) \lor (y_n - y) = d_{st}(x_n, x) \lor d_{st}(y_n, y) \\
\leq \epsilon \lor \epsilon = \epsilon
\]

by Theorem 2.3.8(2) and Lemma 2.3.6(2). Since \( \epsilon > 0 \) is arbitrary, the result follows.

3. Let \( \epsilon > 0 \). Since \( \{x_n\}_{n \in N} \) and \( \{y_n\}_{n \in N} \) are convergent, there exists \( K_1 \in N \) such that for all \( n \geq K_1 \),

\[
d_{st}(x_n, x) = x_n - x \leq \epsilon
\]
and there exists $K_2 \in N$ such that for all $n \geq K_2$,

$$d_{st}(y_n, y) = y_n - y \leq \epsilon.$$ 

Since $N$ is a directed set, there is an element $K \in N$ such that $K_1 \leq K$ and $K_2 \leq K$. Then for all $n \geq K$, we have

$$d_{st}(x_n \wedge y_n, x \wedge y) = (x_n \wedge y_n) - (x \wedge y)$$
$$= ((x_n \wedge y_n) - x) \vee ((x_n \wedge y_n) - y)$$
$$\leq (x_n - x) \vee (y_n - y)$$
$$= d_{st}(x_n, x) \vee d_{st}(y_n, y)$$
$$\leq \epsilon \vee \epsilon = \epsilon$$

by Theorem 2.3.8(1) and Lemma 2.3.6(1). Since $\epsilon > 0$ is arbitrary, the result follows.

\[ \square \]

**Theorem 4.2.2.** Let $V$ be a value quantale. If in $V_{st}$ the net $\{x_n\}_{n \in N}$ converges to $x$ and the net $\{y_n\}_{n \in N}$ converges to $y$, then the net $\{x_n - y_n\}_{n \in N}$ converges to $x - y$.

**Proof.** Let $\epsilon > 0$. By Lemma 2.3.19, there exists a $\delta > 0$ such that $\delta + \delta \leq \epsilon$. Then there exists $K_1 \in N$ such that for all $n \geq K_1$,

$$d_{st}^*(x_n, x) = (x_n - x) \vee (x - x_n) \leq \delta$$

and there exists $K_2 \in N$ such that for all $n \geq K_2$,

$$d_{st}^*(y_n, y) = (y_n - y) \vee (y - y_n) \leq \delta.$$ 

Since $N$ is a directed set, there is an element $K \in N$ such that $K_1 \leq K$ and $K_2 \leq K$. Then for all $n \geq K$, we need to show that

$$d_{st}^*(x_n - y_n, x - y) = [(x_n - y_n) - (x - y)] \vee [(x - y) - (x_n - y_n)] \leq \epsilon.$$ 

By Proposition 2.2.19(2), it is equivalent to show

$$(x_n - y_n) - (x - y) \leq \epsilon.$$
and
\[(x - y) - (x_n - y_n) \leq \epsilon\]
to complete the proof. For the first inequality, we have
\[
(x_n - y_n) - (x - y) \leq [(x_n - x) + (x - y)] - (x - y) \\
\leq (x_n - x) + (y - y_n) \\
\leq [(x_n - x) \vee (x_n - x)] + [(y_n - y) \vee (y - y_n)] \\
\leq \delta + \delta \leq \epsilon
\]
by Theorem 2.3.7. For the second inequality, the argument follows similarly as above to obtain
\[
(x - y) - (x_n - y_n) \leq [(x - x_n) \vee (x_n - x)] + [(y_n - y) \vee (y - y_n)] \\
\leq \delta + \delta \leq \epsilon.
\]
Since \(\epsilon > 0\) is arbitrary, the result follows.

\[\square\]

Remark 4.2.3. Since Theorem 4.2.1 holds in \(V_{st}\), it also holds in \(V^\text{op}_{st}\), \(V^*_s\) and \(V^+_s\). In comparison, Theorem 4.2.2 only holds in \(V^*_s\) and does not hold in \(V_{st}\) since it is not symmetric. If \(V^*_s\) is separated, then the limits in Theorems 4.2.1 and 4.2.2 are all unique.

Notice that if \(V = [0, \infty]\) and \(V^*_s\) is separated, then Theorems 4.2.1 and 4.2.2 are the generalized version of the limit theorems in Real Analysis [2]. (Of course, we need to also replace nets with sequences to obtain the exact results).

Theorem 4.2.4. Let \(V\) be a value quantale and \(\{a_n\}_{n \in N}\) be a convergent net in \(V_{st}\). If \(\lim_{n \in N} a_n = a\) and \(x_n \leq a_n\) for all \(n \in N\), then \(\{x_n\}_{n \in N}\) is also convergent and \(\lim_{n \in N} x_n = a\).

Proof. Let \(\epsilon > 0\). Then there exists a \(K \in N\) such that for every \(n \geq K\),
\[
d_{st}(a_n, a) = a_n - a \leq \epsilon.
\]
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Then for the same $K$, by Lemma 2.3.6 (1), we obtain

$$x_n - a \leq a_n - a \leq \epsilon.$$ 

Since $\epsilon > 0$ is arbitrary, $\lim_{n \in N} x_n = a$. 

Corollary 4.2.5. Let $V$ be a value quantale and $\{b_n\}_{n \in N}$ a convergent net in $V^\text{op}_{st}$. If $\lim_{n \in N} b_n = b$ and $b_n \leq x_n$ for all $n \in N$, then $\{x_n\}_{n \in N}$ is also convergent and $\lim_{n \in N} x_n = b$.

Proof. This is the dual statement of Theorem 4.2.4, thus always holds by Theorem 2.1.22.

Corollary 4.2.6. Let $V$ be a value quantale and, $\{a_n\}_{n \in N}$ and $\{b_n\}_{n \in N}$ be convergent nets in $V^\text{st}_{st}$. If

$$\lim_{n \in N} a_n = \lim_{n \in N} b_n = L$$

and $a_n \leq x_n \leq b_n$ for all $n \in N$, then $\{x_n\}_{n \in N}$ is also convergent and $\lim_{n \in N} x_n = L$.

Proof. This result follows from Theorem 4.2.4 and Corollary 4.2.5.

Remark 4.2.7. Corollary 4.2.6 is the generalized version of the well-known Squeeze Lemma in Real Analysis [2]. Notice that symmetry plays a crucial part for the result above.

Theorem 4.2.8. Let $V$ be a value quantale, $(X, d_X)$ and $(Y, d_Y)$ be $V$-spaces and $\sigma$ is a sieve of $V$. Then the following are equivalent:

1. The function $f : X \to Y$ is continuous at $c$.
2. For every net $\{x_n\}_{n \in N}$ indexed by a directed set $N$, if $\{x_n\}_{n \in N}$ converges to $c$, then the net $\{f(x_n)\}_{n \in N}$ converges to $f(c)$.
3. For every net $\{x_v\}_{v \in \sigma}$, if $\{x_v\}_{v \in \sigma}$ converges to $c$, then the net $\{f(x_v)\}_{v \in \sigma}$ converges to $f(c)$.

Proof. (1) $\implies$ (2) Assume that $f : X \to Y$ is continuous at $c$ and $\lim_{n \in N} x_n = c$. Let $\epsilon > 0$. By definition, there exists a $\delta > 0$ for $c$ such that if $d_X(x, c) \leq \delta$, then

$$d_Y(f(x), f(c)) \leq \epsilon.$$
Since the \( \{x_n\}_{n \in \mathbb{N}} \) is convergent, there exists a \( K \in \mathbb{N} \) such that for all \( n \geq K \), \( d_X(x_n, c) \leq \delta \). Then for the net \( \{f(x_n)\}_{n \in \mathbb{N}} \), we obtain for all \( n \geq K \),

\[
d_Y(f(x_n), f(c)) \leq \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, the net \( \{f(x_n)\}_{n \in \mathbb{N}} \) converges to \( f(c) \).

\[ (2) \implies (3) \text{ Obvious.} \]

\[ (3) \implies (1) \text{ Suppose } f \text{ is not continuous at } c. \text{ This means that there exists an } \epsilon_0 > 0 \text{ such that for every } \delta > 0, \text{ there is an } x \in X \text{ such that } d(x, c) \leq \delta \text{ but } d(f(x), f(c)) \not\leq \epsilon_0. \text{ Hence for every } s \in \sigma, \text{ we choose an } x_s \in X \text{ such that } d(x_s, c) \leq s \text{ but } d(f(x_s), (f(c)) \not\leq \epsilon_0, \]

thus the net \( \{f(x_s)\}_{s \in \sigma} \) does not converge to \( f(c) \). We claim that \( \{x_s\}_{s \in \sigma} \) converges to \( c \). Given \( \epsilon > 0 \), take any \( s \in \sigma \) such that \( s \leq \epsilon \). Then for every \( t \geq_{op} s \) (or equivalently, \( t \leq s \)), we have

\[
d(x_t, c) \leq t \leq s \leq \epsilon,
\]

thus \( \{x_s\}_{s \in \sigma} \) converges to \( c \), finishing the proof. \( \square \)

Remark 4.2.9. The result above generalizes the well-known theorem pertaining to the equivalence of Heine’s sequential definition \([19, 27]\) and Cauchy’s \( \epsilon - \delta \) definition of continuity in Real Analysis \([2]\).

### 4.3 Monotone Nets

In this section, we define monotone nets in \( V_{st} \) and present some results on the convergence of monotone nets.

**Definition 4.3.1.** Let \( V \) be a value quantale. Then \( \{x_n\}_{n \in \mathbb{N}} \) is a *monotonically increasing net* in \( V_{st} \) if for every \( m \geq n \) in \( \mathbb{N} \), \( x_m \geq x_n \). \( \{x_n\}_{n \in \mathbb{N}} \) is a *monotonically decreasing net* in \( V_{st} \) if for every \( m \geq n \) in \( \mathbb{N} \), \( x_m \leq x_n \).

**Proposition 4.3.2.** Let \( V \) be a value quantale. If \( \{x_n\}_{n \in \mathbb{N}} \) is monotonically increasing in \( V_{st} \) and \( \bigvee x_n = x \), then \( x_n \) converges to \( x \).

**Proof.** Let \( \epsilon > 0 \). By Proposition 2.2.19(4), \( x = \bigvee x_n \) implies that \( x_n \leq x \) for every \( n \in \mathbb{N} \), then by Theorem 2.3.7(1) and Proposition 2.2.35(1), we obtain

\[
d_{st}(x_n, x) = x_n - x = 0 \leq \epsilon.
\]
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Since $\epsilon \succ 0$ is arbitrary, $x_n$ converges to $x$. \hfill $\square$

Remark 4.3.3. Note that the limit of the net in Proposition 4.3.2 is not unique. In fact, all the elements $y \geq \bigvee x_n$ are limits of the monotonically increasing net since $x_n \leq x \leq y$ which implies

$$x_n - y = 0 \leq \epsilon.$$ 

Theorem 4.3.4. Let $V$ be a value quantale and $\{x_n\}_{n \in N}$ a monotonically decreasing net in $V_{st}$. If $\bigwedge_{n \in N} x_n = x$ and

$$x + \epsilon \succ x$$

for every $\epsilon \succ 0$ in $V_{st}$, then $\{x_n\}_{n \in N}$ converges to $x$.

Proof. Let $\epsilon \succ 0$. We need to show that there exists a $K \in N$ such that for every $n \geq K$, $x_n - x \leq \epsilon$. Since $\bigwedge_{n \in N} x_n = x$,

$$x \succ \bigwedge x_n$$

which by definition of the well-above relation implies that there exists a $K \in N$ such that

$$x_K \leq x + \epsilon.$$ 

For every $n \geq K$, $x_n \leq x_K$ since the net is monotonically decreasing, so we obtain

$$x_n \leq x + \epsilon$$

which is equivalent to $d_{st}(x_n, x) = x_n - x \leq \epsilon$. Since $\epsilon \succ 0$ is arbitrary, the result follows. \hfill $\square$
Chapter 5

The Value Quantale of Distance Distribution Functions

This chapter is dedicated to presenting a particular value quantale known as $\Delta$. The first section includes a detailed version of the underlying structures of the value distributive lattice of distance distribution functions given by Flagg [15]. Then in the next section, we introduce an addition operation turning $\Delta$ into a quantale. Such a detailed verification appears to be new in the literature.

5.1 The Value Distributive Lattice of Distance Distribution Functions

We define and verify the underlying structures of the value distributive lattice of distance distribution functions except the addition operation, which we defer to the next section.

Definition 5.1.1. Let $M$ be the set of monotone maps $F: [0, \infty) \to [0, 1]$ with the point-wise ordering, namely given two functions $F, G \in M$, we have $F \leq G$ if and only if $F(x) \leq G(x)$, for every $x \in [0, \infty)$. Note that the maps $F_0, F_1 \in M$ where $F_0(x) = 0$ and $F_1(x) = 1$ for all $x \in [0, \infty)$.

Proposition 5.1.2. The set $M$ with the ordering $\leq$ given above is a poset.

Proof. Trivial. $\square$

Proposition 5.1.3. The set $M$ is a complete lattice with $\bigvee$ and $\bigwedge$ computed point-wise.

Proof. By Lemma 2.2.30, it suffices to show that for every $S \subseteq M$, $\bigwedge S$ exists in $M$. The existence of a lower bound of $S$ is guaranteed since for all $G \in S$, $G(x) \geq F_0(x)$
for all \( x \in [0, \infty) \) which means that \( G \geq F_0 \) for all \( G \in S \). So we show the existence of \( \bigwedge S \) in two parts: (1) for any nonempty subset \( S \subseteq M \) and (2) for \( \emptyset \subset M \).

1. Let \( S \subseteq M \) be nonempty. We construct the map \( \bigwedge S \) by

\[
\left( \bigwedge S \right)(x) = \bigwedge_{G \in S} G(x) \text{ for all } x \in [0, \infty).
\]

Now we verify three things:

(a) \( \bigwedge S \) is map from \([0, \infty)\) to \([0, 1]\): Let \( x \in [0, \infty) \). Since for all \( G \in S \), \( 0 \leq G(x) \leq 1 \), then \( \bigwedge_{G \in S} G(x) \) is guaranteed by the completeness of \([0, 1]\), thus

\[
0 \leq \bigwedge_{G \in S} G(x) \leq 1,
\]

or simply \( 0 \leq (\bigwedge S)(x) \leq 1 \). Since \( x \in [0, \infty) \) is arbitrary, it follows that \( \bigwedge S \) is a map from \([0, \infty)\) to \([0, 1]\).

(b) \( \bigwedge S \) is monotone: Let \( x \leq y \) where \( x, y \in [0, \infty) \). Since all maps in \( M \) are monotone and \( S \subseteq M \), \( G(x) \leq G(y) \) for every \( G \in S \). We need to show that

\[
\left( \bigwedge S \right)(x) = \bigwedge_{K \in S} K(x) \leq \bigwedge_{G \in S} G(y) = \left( \bigwedge S \right)(y),
\]

or equivalently, \( \bigwedge_{K \in S} K(x) \leq G(y) \) for every \( G \in S \), by Proposition 2.2.19(1) which indeed holds since

\[
\bigwedge_{K \in S} K(x) \leq G(x) \leq G(y).
\]

Since \( x, y \in [0, \infty) \) are arbitrary, \( \bigwedge S \) is monotone. Thus it immediately follows from (a) and (b) that \( \bigwedge S \in M \).

(c) \( \bigwedge S \) is the infimum of \( S \): We can easily see that \( \bigwedge S \) is a lower bound of \( S \) since for every \( x \in [0, \infty) \),

\[
\left( \bigwedge S \right)(x) = \bigwedge_{G \in S} G(x) \leq H(x)
\]

for all \( H \in S \). Now given a lower bound \( L \) of \( S \), we know that \( L \leq G \) for every \( G \in S \). This means that for every \( x \in [0, \infty) \), \( L(x) \leq G(x) \) for every \( G \in S \). Then by Proposition 2.2.19(1),

\[
L(x) \leq \bigwedge_{G \in S} G(x) = \left( \bigwedge S \right)(x)
\]
for every $x \in [0, \infty)$ which means $L \leq \bigwedge S$, thus $\bigwedge S$ is the infimum of $S$.

2. Let $\emptyset \subseteq M$. By Proposition 2.2.19(3), we have

$$\bigwedge S \leq \bigwedge \emptyset$$

for any $S \subseteq M$. For any $F \in M$, the above inequality implies that $F \leq \bigwedge \emptyset$ and since $F \leq F_1$ holds, we assign $\bigwedge \emptyset = F_1$.

\[ \square \]

**Definition 5.1.4.** Let $F \in M$ be a monotone map. Then $F$ is called a *distance distribution function* (or simply *d.d.f.*) if $F$ is left-continuous:

$$\forall x \in [0, \infty), \bigvee_{y<x} F(y) = F(x).$$

We denote $\Delta \subseteq M$ the set of all d.d.f.’s, i.e.,

$$\Delta = \{ F : [0, \infty) \rightarrow [0, 1] \mid \bigvee_{y<x} F(y) = F(x), \forall x \in [0, \infty) \}.$$  

The set $\Delta$ is endowed with the opposite of the point-wise ordering given by $F \leq_{op} G$ if and only if for all $x \in [0, \infty)$, we have $F(x) \geq G(x)$.

**Remark 5.1.5.** Note that every step-function $F_{a,b}$ is in $\Delta$ where

$$F_{a,b} = \begin{cases} 0, & x \leq a \\ b, & x > a \end{cases}$$

for $a \in [0, \infty)$ and $0 \leq b \leq 1$.

**Proposition 5.1.6.** The pair $(\Delta, \leq_{op})$ is a poset.

**Proof.** Trivial. 

\[ \square \]

**Proposition 5.1.7.** For any function $F \in \Delta$, it holds that $F(0) = 0$.

**Proof.** For any $F \in \Delta$, $F$ is left continuous, thus we obtain

$$F(0) = \bigvee_{y<0} F(y) = \bigvee \emptyset = 0,$$

from Remark 2.2.11, where $\emptyset \subset [0, 1]$.

\[ \square \]
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The bottom element of \((\Delta, \leq_{op})\) is
\[
F_{0,1}(x) = \begin{cases} 
0, & x = 0 \\
1, & \text{otherwise} 
\end{cases}
\]
and the top element is \(F_0(x) = 0\) for every \(x \in [0, \infty)\).

**Proposition 5.1.8.** The poset \((\Delta, \leq_{op})\) is a complete lattice.

**Proof.** By Lemma 2.2.30, it suffices to show that for every \(S \subseteq \Delta\), \(H = \bigwedge S\) exists in \(\Delta\). \(H\) is simply defined by
\[
H(x) = \bigvee_{G \in S} G(x).
\]
The proof is given in two parts: for (1) non-empty \(S \subseteq \Delta\) and (2) \(S = \emptyset\).

1. Let \(S \subseteq \Delta\) be non-empty and \(x \in [0, \infty)\).

   (a) We begin by showing \(H \in \Delta\), i.e., \(\bigvee_{y \lessdot x} H(y) = H(x)\), or equivalently,
\[
\bigvee \left( \bigvee_{F \in S} F(y) \right) = \bigvee G(x).
\]

Then it suffices to show that
\[
\bigvee_{y \lessdot x} \left( \bigvee_{F \in S} F(y) \right) \leq \bigvee_{G \in S} G(x)
\]
and
\[
\bigvee_{y \lessdot x} \left( \bigvee_{F \in S} F(y) \right) \geq \bigvee_{G \in S} G(x).
\]

For the first inequality
\[
\bigvee_{y \lessdot x} \left( \bigvee_{F \in S} F(y) \right) \leq \bigvee_{G \in S} G(x),
\]
by Proposition 2.2.19(2), it is equivalent to show that \(F(y) \leq \bigvee_{G \in S} G(x)\) for all \(F \in S\) and \(y < x\) which indeed holds since
\[
F(y) \leq F(x) \leq \bigvee_{G \in S} G(x).
\]
For the second inequality, we need to show
\[ \bigvee_{y<x} \left( \bigvee_{F \in S} F(y) \right) \geq \bigvee_{G \in S, z<x} G(z), \]
and it holds if
\[ \bigvee_{y<x} \bigvee_{F \in S} F(y) \geq G(z) \]
for every \( G \in S \) and \( z < x \) by Proposition 2.2.19(2), which indeed holds since
\[ G(z) \leq \bigvee_{F \in S} F(z) \leq \bigvee_{y<x} \bigvee_{F \in S} F(y). \]

(b) Now we show that \( H \) is a lower bound of \( S \). Let \( F \in S \). It is easy to see that \( H(x) \geq F(x) \) for all \( x \in [0, \infty) \) since
\[ \bigvee_{G \in S} G(x) \geq F(x), \]
thus \( H \leq_{op} F \). Since \( F \in S \) is arbitrary, \( H \) is a lower bound of \( S \).

(c) Next we show that \( H \) is the greatest lower bound of \( S \). Let \( H' \) be a lower bound of \( S \). So \( H' \leq_{op} F \) for all \( F \in S \), i.e., \( H'(x) \geq F(x) \) for every \( x \in [0, \infty) \) and \( F \in S \). We need to show that \( H' \leq_{op} H \), i.e., \( H'(x) \geq H(x) \), for every \( x \in [0, \infty) \), or equivalently,
\[ H'(x) \geq \bigvee_{F \in S} F(x). \]
By Proposition 2.2.19(2), the inequality above is equivalent to
\[ H'(x) \geq F(x) \]
for every \( F \in S \) and \( x \in [0, \infty) \) which indeed holds by the assumption. Since \( H' \) is an arbitrary lower bound of \( S \), \( H \) is the greatest lower bound of \( S \).

2. Now, we show \( \bigwedge \emptyset \) exists in \( \Delta \). Since \( \emptyset \subseteq T \) for every \( T \subseteq \Delta \),
\[ \bigwedge S \leq_{op} \bigwedge \emptyset \]
by Proposition 2.2.19(3). For any \( F \in \Delta \), the inequality above implies that \( F \leq_{op} \bigwedge \emptyset \) and since \( F \leq_{op} F_0 \) holds, we assign \( \bigwedge \emptyset = F_0 \).
For $0 \leq \epsilon \leq 1$ and $0 \leq \delta$, let $F_{\delta,1-\epsilon}$ be the d.d.f. defined by
\[
F_{\delta,1-\epsilon}(x) = \begin{cases} 
0, & 0 \leq x \leq \delta, \\
1 - \epsilon, & \delta < x.
\end{cases}
\]

**Proposition 5.1.9.** For any $H \in (\Delta, \leq_{op})$ and all $0 < \epsilon < 1$ and $0 < \delta$,
1. if $H(\delta) > F_{\delta,1-\epsilon}(x) = 1 - \epsilon$ for $x > \delta$, then $F_{\delta,1-\epsilon} \succ H$ and
2. $H \leq_{op} F_{\delta,1-\epsilon}$ if and only if for all $x > \delta$, $H(x) \geq 1 - \epsilon$.

**Proof.**

1. We need to show that $F_{\delta,1-\epsilon} > H$. Let $S \subseteq \Delta$ with $\bigwedge S \leq_{op} H$. This means that
\[
H(x) \leq \bigvee_{G \in S} G(x)
\]
for every $x \in [0, \infty)$. Suppose that for $x > \delta$, $F_{\delta,1-\epsilon}(x) = 1 - \epsilon < H(\delta)$. Then
\[
F_{\delta,1-\epsilon}(x) = 1 - \epsilon < H(\delta) \leq \bigvee_{G \in S} G(\delta).
\]
Since $1 - \epsilon < \bigvee_{G \in S} G(\delta)$, there exists an $L \in S$ such that
\[
1 - \epsilon < L(\delta) \leq L(x)
\]
for $x > \delta$, which implies that $F_{\delta,1-\epsilon} \leq L$ or equivalently $L \leq_{op} F_{\delta,1-\epsilon}$.

2. $H \leq_{op} F_{\delta,1-\epsilon}$ is equivalent to
\[
H(x) \geq F_{\delta,1-\epsilon}(x)
\]
for all $x \in [0, \infty)$ which is equivalent to $H(x) \geq 1 - \epsilon$ for every $x > \delta$.

**Proposition 5.1.10.** For $G \in (\Delta, \leq_{op})$, the following are equivalent:

1. $G \succ F_{0,1}$
2. $F_{\epsilon,1-\epsilon} \leq_{op} G$ for some $\epsilon \in (0, 1]$.
3. There exists an $\epsilon \in (0, 1]$ such that $G(x) \leq F_{\epsilon,1-\epsilon}(x)$ for $x > \epsilon$. 

Proof. (1) $\implies$ (2). Suppose that $G \succ F_{0,1}$. Take the collection $\{F_{\epsilon,1-\epsilon} \mid 0 < \epsilon \leq 1\}$. We need to show that

$$\bigwedge_{0<\epsilon\leq1} F_{\epsilon,1-\epsilon} = F_{0,1}. $$

By the construction given in the proof of Proposition 5.1.8,

$$\left( \bigwedge_{0<\epsilon\leq1} F_{\epsilon,1-\epsilon}\right)(x) = \bigvee_{0<\epsilon\leq1} (F_{\epsilon,1-\epsilon}(x))$$

for every $x \in (0, \infty)$. Then we obtain, for $x > 0$

$$\bigvee_{0<\epsilon\leq1} (F_{\epsilon,1-\epsilon}(x)) = \bigvee_{0<\epsilon\leq1} (1 - \epsilon) = 1$$

which is exactly $F_{0,1}$. Then by definition of the well-above relation $\succ$, there exists an $\epsilon \in (0, 1]$ such that $F_{\epsilon,1-\epsilon} \leq_{op} G$.

(2) $\implies$ (3). Obvious.

(3) $\implies$ (1). Suppose there exists an $\epsilon \in (0, 1]$ such that $G(x) \leq F_{\epsilon,1-\epsilon}(x)$ for $x > \epsilon$. Since $\epsilon > 0$, then for every $x > \epsilon$,

$$F_{0,1}(x) = 1 > 1 - \epsilon = F_{\epsilon,1-\epsilon}(x),$$

thus $F_{0,1}(x) > F_{\epsilon,1-\epsilon}(x)$ for $x > \epsilon$. This implies that $F_{\epsilon,1-\epsilon} \succ F_{0,1}$ by Proposition 5.1.9(1). Since $F_{\epsilon,1-\epsilon} \leq_{op} G$, by Proposition 2.2.35(3), we obtain $G \succ F_{0,1}$. 

We will now show that $(\Delta, \leq_{op})$ is completely distributive and also a value distributive lattice.

**Proposition 5.1.11.** The complete lattice $(\Delta, \leq_{op})$ is completely distributive.

**Proof.** Given $H \in \Delta$, by Theorem 2.2.37, we need to show that for

$$T = \{G \in \Delta \mid G \succ H\},$$

$$\bigwedge T = H$$

or equivalently, $\bigwedge T \geq_{op} H$ and $\bigwedge T \leq_{op} H$. The first inequality $\bigwedge T \geq_{op} H$ indeed holds since $G \geq_{op} H$ for every $G \in T$ by Propositions 2.2.19(1) and 2.2.35(1). For the second inequality $\bigwedge T \leq_{op} H$, we utilize the set

$$S = \{F_{\delta,1-\epsilon} \mid F_{\delta,1-\epsilon} \succ H, 0 < \epsilon < 1, 0 < \delta\}.$$ 

Clearly, $S \subseteq T$, then $\bigwedge T \leq_{op} \bigwedge S$ by Proposition 2.2.19(3), thus it suffices to show that $\bigwedge S = H$. Notice that $H$ is a lower bound of $S$ since for every $F_{\delta,1-\epsilon} \in S$,
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$F_{\delta,1-\epsilon} \geqop H$ implies that $F_{\delta,1-\epsilon} \geqop H$ by Proposition 2.2.35(1). Next, to show that $H$ is the greatest lower bound of $S$, we now only need to show that $H_{op} \geq \bigwedge S$, that is,

$$H(x) \leq \bigvee_{G \in S} G(x) \forall x.$$  

For this, fix $x$, and we show that $H(x) \leq \bigvee_{G \in S} G(x)$. If $H(x) = 0$, then we are done, so assume $H(x) > 0$ (hence $x > 0$). Fix $\epsilon'$ such that $0 < \epsilon' < 2H(x)$. We will find a $G \in S$ such that

$$H(x) \leq G(x) + \epsilon'.$$

Since $H$ is left-continuous at $x$,

$$H(x) = \bigvee_{t < x} H(t),$$

so there exists a $0 < \delta < x$ such that $H(\delta) > H(x) - \frac{\epsilon'}{2}$. Let $\epsilon = 1 + \frac{\epsilon'}{2} - H(x)$. Then $0 < \epsilon < 1$. Further, $G = F_{\delta,1-\epsilon} \geqop H$ by Proposition 2.2.19(1) and $G$ satisfies $H(x) \leq G(x) + \epsilon'$.

\[ \square \]

**Proposition 5.1.12.** The complete lattice $(\Delta, \leq_{op})$ is a value distributive lattice.

*Proof.* We need to show that for $G \geq F_{0,1}$ and $H \geq F_{0,1}$, we have $G \wedge H \geq F_{0,1}$. By Proposition 5.1.10, it suffices to show that there exists an $\epsilon \in (0,1]$ such that

$$G(x) \vee H(x) \leq F_{\epsilon,1-\epsilon}(x)$$

for $\forall x > \epsilon$. By the same proposition, we know there are $\epsilon', \epsilon'' \in (0,1]$ such that

$$G(x) \leq F_{\epsilon',1-\epsilon'}(x) = 1 - \epsilon'$$

and

$$H(x) \leq F_{\epsilon'',1-\epsilon''}(x) = 1 - \epsilon'',$$

respectively. We know $\epsilon' \wedge \epsilon'' > 0$ so take $\epsilon = \epsilon' \wedge \epsilon''$. Since

$$1 - \epsilon' \leq 1 - (\epsilon' \wedge \epsilon'') = 1 - \epsilon$$

and

$$1 - \epsilon'' \leq 1 - (\epsilon' \wedge \epsilon'') = 1 - \epsilon,$$
we obtain \( G(x) \leq 1 - \epsilon \) and \( H(x) \leq 1 - \epsilon \). Then, by Proposition 2.2.19(2),

\[
G(x) \lor H(x) \leq 1 - \epsilon = F_{\epsilon,1-\epsilon}(x),
\]

thus \( G \land H \succ F_{\epsilon,1-\epsilon} \).

\[\square\]

5.2 The Value Quantale of Distance Distribution Functions

In this section, we introduce an addition operation on \((\Delta, \leq_{op})\) to obtain the value quantale \((\Delta, \leq_{op}, +)\). Such an explicit definition and detailed verification of an addition operation on \((\Delta, \leq_{op}, +)\) appears not to be found in the current literature. We define the addition operation on \((\Delta, \leq_{op})\) by

\[
(G + H)(x) = \bigvee_{0 \leq t \leq x} G(t)H(x - t)
\]

for every \( x \in [0, \infty) \).

The following proposition discusses some important properties of the addition operation.

**Proposition 5.2.1.** Let \( G, H \) be functions in \( M \). Then the following are true.

1. \( G + H \) is monotone.
2. For \( G, H \in (\Delta, \leq_{op}) \), it holds that

\[
\bigvee_{y < x} \left( \bigvee_{0 \leq t \leq y} G(t)H(y - t) \right) = \bigvee_{0 \leq u \leq x} \left( \bigvee_{z < x} G(u)H(z - u) \right).
\]

**Proof:**

1. Let \( y \leq x \) for \( x, y \in [0, \infty) \). We will show that \((G + H)(y) \leq (G + H)(x)\), i.e.,

\[
\bigvee_{0 \leq t \leq y} G(t)H(y - t) \leq \bigvee_{0 \leq u \leq x} G(u)H(x - u).
\]

By Proposition 2.2.19(2), it suffices to show that for every \( 0 \leq t \leq y \),

\[
G(t)H(y - t) \leq \bigvee_{0 \leq u \leq x} G(u)H(x - u).
\]

Fix \( t \). Since \( H \) is monotone, we have

\[
H(y - t) \leq H(x - t),
\]
then

\[ G(t)H(y - t) \leq G(t)H(x - t). \]

Since \( t \leq y \leq x \),

\[ G(t)H(y - t) \leq G(t)H(x - t) \leq \bigvee_{0 \leq u \leq x} G(u)H(x - u). \]

Thus

\[ G(t)H(y - t) \leq \bigvee_{0 \leq u \leq x} G(u)H(x - u). \]

Since \( 0 \leq t \leq y \) is arbitrary, \( G + H \) is monotone.

2. We need to show that

\[ \bigvee_{y < x} \left( \bigvee_{0 \leq t \leq y} G(t)H(y - t) \right) \leq \bigvee_{0 \leq u \leq x} \left( \bigvee_{z < x} G(u)H(z - u) \right) \]

and

\[ \bigvee_{y < x} \left( \bigvee_{0 \leq t \leq y} G(t)H(y - t) \right) \geq \bigvee_{0 \leq u \leq x} \left( \bigvee_{z < x} G(u)H(z - u) \right). \]

We begin by noting that

\[ \bigvee_{0 \leq u \leq x} \left( \bigvee_{z < x} G(u)H(z - u) \right) = \bigvee_{0 \leq u \leq x} G(u) \left( \bigvee_{z < x} H(z - u) \right) \]

\[ = \bigvee_{0 \leq u \leq x} G(u)H(x - u) \]

since \( H \) is left continuous. Thus simplifying the first inequality, we need to show that

\[ \bigvee_{y < x} \left( \bigvee_{0 \leq t \leq y} G(t)H(y - t) \right) \leq \bigvee_{0 \leq u \leq x} G(u)H(x - u). \]

By Proposition 2.2.19(2), the above is equivalent to

\[ \bigvee_{0 \leq t \leq y} G(t)H(y - t) \leq \bigvee_{0 \leq u \leq x} G(u)H(x - u) \]

for every \( y < x \), which always holds by part (1) above. The second inequality

\[ \bigvee_{y < x} \left( \bigvee_{0 \leq t \leq y} G(t)H(y - t) \right) \geq \bigvee_{0 \leq u \leq x} \left( \bigvee_{z < x} G(u)H(z - u) \right) \]
holds if and only if, for every $0 \leq u \leq x$ and $z < x$,

$$
\bigvee_{y < x} \left( \bigvee_{0 \leq t \leq y} G(t)H(y - t) \right) \geq G(u)H(z - u)
$$

by Proposition 2.2.19(2). Fix $u$ and $z$ such that $0 \leq u \leq x$ and $z < x$. So there are two cases to consider: $u \leq z$ and $u \geq z$. If $u \leq z$, then

$$
u \leq z < x
$$

thus

$$
G(u)H(z - u) \leq \bigvee_{0 \leq t \leq z} G(t)H(z - t)
$$

$$
\leq \bigvee_{y < x} \left( \bigvee_{0 \leq t \leq y} G(t)H(y - t) \right)
$$

by monotonicity in part (1) above. For the second case, if $u \geq z$, then we have to show that

$$
\bigvee_{y < x} \left( \bigvee_{0 \leq t \leq y} G(t)H(y - t) \right) \geq G(u)H(z - u)
$$

which holds trivially since $z - u = \max\{0, z - u\} = 0$ and $H(z - u) = H(0) = 0$ by Proposition 5.1.7. Since $u$ and $z$ are arbitrary, the result follows.

\[\]
1. Notice that
\[
\bigvee_{0 \leq u \leq x} \left( \bigvee_{z < x} G(u)H(z - u) \right) = \bigvee_{0 \leq u \leq x} G(u) \left( \bigvee_{z < x} H(z - u) \right) \\
= \bigvee_{0 \leq u \leq x} G(u)H(x - u) \\
= (G + H)(x)
\]
since $H$ is left continuous. By Proposition 5.2.1(2), we obtain
\[
\bigvee_{y < x} (G + H)(y) = \bigvee_{y < x} \left( \bigvee_{0 \leq t \leq y} G(t)H(y - t) \right) \\
= \bigvee_{0 \leq u \leq x} \left( \bigvee_{z < x} G(u)H(z - u) \right) \\
= (G + H)(x).
\]
Since $x \in [0, \infty)$ is arbitrary, $G + H \in \Delta$.

2. Next, we show that $(G + H)(x) = (H + G)(x)$. Given
\[
(G + H)(x) = \bigvee_{0 \leq t \leq x} G(t)H(x - t),
\]
denote $u = x - t$, then since $0 \leq t \leq x$ which means $0 \leq x - t \leq x$, we have $0 \leq u \leq x$, thus
\[
\bigvee_{0 \leq t \leq x} G(t)H(x - t) = \bigvee_{0 \leq u \leq x} G(x - u)H(u) \\
= \bigvee_{0 \leq u \leq x} H(u)G(x - u) \\
= (H + G)(x).
\]
Since $x \in [0, \infty)$ is arbitrary, $G + H = G + H$.

3. Next, we show that for $x \in [0, \infty)$, $((G + H) + K)(x) = (G + (H + K))(x)$. 


Note that
\[
((G + H) + K)(x) = \bigvee_{0 \leq r \leq x} (G + H)(r)K(x - r)
\]
\[
= \bigvee_{0 \leq r \leq x} \left( \bigvee_{0 \leq s \leq r} G(s)H(r - s)K(x - r) \right)
\]
\[
= \bigvee_{0 \leq s \leq r \leq x} G(s)H(r - s)K(x - r)
\]
and
\[
(G + (H + K))(x) = \bigvee_{0 \leq t \leq x} G(t)(H + K)(x - t)
\]
\[
= \bigvee_{0 \leq t \leq x} \left( \bigvee_{0 \leq u \leq x - t} G(t)H(u)K(x - t - u) \right)
\]
\[
= \bigvee_{0 \leq t + u \leq x} G(t)H(u)K(x - (t + u)).
\]
For \((G + (H + K))(x)\), denote \(t + u = v\), then
\[
\bigvee_{0 \leq t + u \leq x} G(t)H(u)K(x - (t + u)) = \bigvee_{0 \leq t \leq v \leq x} G(t)H(v - t)K(x - v)
\]
\[
= ((G + H) + K)(x).
\]
Since \(x \in [0, \infty)\) is arbitrary, \(G + (H + K) = (G + H) + K\).

4. We show that \((G + F_{0,1})(x) = G(x)\). We know
\[
(G + F_{0,1})(x) = \bigvee_{0 \leq t \leq x} G(t)F_{0,1}(x - t).
\]
For \(t = 0\) and \(t = x\), we have \(G(0)F_{0,1}(x - 0) = 0\) and \(G(x)F_{0,1}(0) = 0\), respectively. If \(0 < t < x\), then \(F_{0,1}(x - t) = 1\) then we obtain
\[
\bigvee_{0 \leq t \leq x} G(t)F_{0,1}(x - t) = \bigvee_{0 < t < x} G(t)F_{0,1}(x - t)
\]
\[
= \bigvee_{t < x} G(t) \cdot 1
\]
\[
= G(x).
\]
Since \(x \in [0, \infty)\) is arbitrary, \(G + F_{0,1} = G\).
5. Next, we show that for a subset $S \subseteq \Delta$ and $x \in [0, \infty)$,

$$(G + \bigwedge S)(x) = \bigwedge (G + S)(x).$$

Since $\bigwedge S(t) = \bigvee_{H \in S} H(t)$ for every $t \in [0, \infty)$, we have

$$(G + \bigwedge S)(x) = \bigvee_{0 \leq t \leq x} G(t) \bigwedge S(x - t)$$

$$= \bigvee_{0 \leq t \leq x} G(t) \left( \bigvee_{H \in S} H(x - t) \right)$$

$$= \bigvee_{0 \leq t \leq x} \left( \bigvee_{H \in S} G(t)H(x - t) \right)$$

and

$$\bigwedge (G + S)(x) = \bigvee_{K \in S} \left( (G + K)(x) \right)$$

$$= \bigvee_{K \in S} \left( \bigvee_{0 \leq u \leq x} G(u)K(x - u) \right).$$

So we need to show

$$\bigvee_{0 \leq t \leq x} \left( \bigvee_{H \in S} G(t)H(x - t) \right) = \bigvee_{K \in S} \left( \bigvee_{0 \leq u \leq x} G(u)K(x - u) \right)$$

which is equivalent to

$$\bigvee_{0 \leq t \leq x} \left( \bigvee_{H \in S} G(t)H(x - t) \right) \geq \bigvee_{K \in S} \left( \bigvee_{0 \leq u \leq x} G(u)K(x - u) \right)$$

and

$$\bigvee_{0 \leq t \leq x} \left( \bigvee_{H \in S} G(t)H(x - t) \right) \leq \bigvee_{K \in S} \left( \bigvee_{0 \leq u \leq x} G(u)K(x - u) \right).$$

For the first inequality, by Proposition 2.2.19(2), it is enough to show that for every $K \in S$ and every $0 \leq u \leq x$,

$$G(u)K(x - u) \leq \bigvee_{0 \leq t \leq x} \left( \bigvee_{H \in S} G(t)H(x - u) \right).$$
Fix $K$ and $u$, then by Proposition 2.2.19(4), we have

$$
G(u)K(x - u) \leq \bigvee_{H \in S} G(u)H(x - u)
$$

$$
\leq \bigvee_{0 \leq t \leq x} \left( \bigvee_{H \in S} G(t)H(x - t) \right).
$$

Since $K$ and $u$ are arbitrary, the first inequality holds. Also, for the second inequality, by Proposition 2.2.19(2), it is enough to show that for all $0 \leq t \leq x$ and $H \in S$,

$$
G(t)H(x - t) \leq \bigvee_{K \in S} \left( \bigvee_{0 \leq u \leq x} G(u)K(x - u) \right).
$$

Fix $t$ and $H$, then by Proposition 2.2.19(4), we have

$$
G(t)H(x - t) \leq \bigvee_{0 \leq u \leq x} G(u)H(x - u)
$$

$$
\leq \bigvee_{K \in S} \left( \bigvee_{0 \leq u \leq x} G(u)K(x - u) \right).
$$

Since $t$ and $H$ are arbitrary, the second inequality holds.

\[\square\]

With the above result, we confirm that $(\Delta, \leq, +)$ is a value quantale. The example below shows an interesting property of the addition operation.

**Example 5.2.3.** Given two step-functions $H_{a,b}$ and $H_{c,d}$ in $\Delta$, we obtain

$$
H_{a,b} + H_{c,d} = \bigvee_{0 \leq t \leq x} H_{a,b}(t)H_{c,d}(x - t)
$$

which is just another step-function given by

$$
H_{a+c,bd} = \begin{cases} 
0, & x \leq a + c \\
bd, & x > a + c.
\end{cases}
$$
Chapter 6

Cauchy Completion

This chapter presents a new completion construction of a weak form of symmetric $V$-spaces using Cauchy filters. In the first section, we define the property of uniformly vanishing asymmetry and also give details into the notion of balls in a $V$-space. The second section introduces filters and other machinery to be used in the construction. The third section talks about the completeness and op-completeness of $V_st$ and the last section presents the completion construction.

6.1 Uniformly Vanishing Asymmetry

We begin by defining the property of uniformly vanishing asymmetry, a weak form of symmetry and give a few fundamental results on balls in $V$-space.

**Definition 6.1.1.** Let $X$ be a $V$-space and $\epsilon > 0$. Then the sets

$$x_\epsilon = \{ y \in X \mid d(x, y) \leq \epsilon \}$$

and

$$x'_\epsilon = \{ y \in X \mid d(y, x) \leq \epsilon \}$$

are **balls centered at point $x \in X$ with radius $\epsilon > 0$.**

For a subset $S \subseteq X$, we define

$$S_\epsilon = \bigcup_{x \in S} x_\epsilon = \{ y \in X \mid \exists x \in S : d(x, y) \leq \epsilon \}$$

and

$$S'_\epsilon = \bigcup_{x \in S} x'_\epsilon = \{ y \in X \mid \exists x \in S : d(y, x) \leq \epsilon \}$$

be the **union of balls** of all points in $S$ with some fixed radius $\epsilon > 0$. 

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Remark 6.1.2. Since symmetry is not satisfied in a $V$-space $X$, $x = x'$ does not always hold. Also, notice that $x = \{x\}$ and $x' = \{x\}$. In addition, given $x \subseteq X$ and $x' \subseteq X^\op$ for fixed $\epsilon > 0$ and $x \in X$, $x$ and $x'$ are the same set since

$$d(x, y) = d^\op(y, x)$$

for every $y \in X$.

Proposition 6.1.3. Let $X$ be a $V$-space. Then the following hold for every $\epsilon > 0$ and $\epsilon' > 0$.

1. If $\epsilon' \leq \epsilon$, then $x_{\epsilon'} \subseteq x_\epsilon$ and $x_{\epsilon'} \subseteq x_\epsilon$.
2. $(S_\epsilon)_{\epsilon'} \subseteq S_{\epsilon+\epsilon'}$ and $(S')_{\epsilon'} \subseteq S^{\epsilon+\epsilon'}$.

Proof.

1. Suppose $\epsilon' \leq \epsilon$. Let $y \in x_{\epsilon'}$. Then $d(x, y) \leq \epsilon \leq \epsilon'$, thus $y \in x_\epsilon$. Since $y \in x_{\epsilon'}$ is arbitrary, $x_{\epsilon'} \subseteq x_\epsilon$. The argument for $x_{\epsilon'} \subseteq x_\epsilon$ follows similarly.

2. Let $z \in (S_\epsilon)_{\epsilon'}$. This means $d(y, z) \leq \epsilon'$ for some $y \in S_\epsilon$, thus $d(x, y) \leq \epsilon$ for some $x \in S$. Then $d(x, z) \leq d(x, y) + d(y, z) \leq \epsilon + \epsilon'$, thus $z \in S_{\epsilon+\epsilon'}$. Since $z \in (S_\epsilon)_{\epsilon'}$ is arbitrary, $(S_\epsilon)_{\epsilon'} \subseteq S_{\epsilon+\epsilon'}$. The argument for $(S')_{\epsilon'} \subseteq S^{\epsilon+\epsilon'}$ follows similarly.

\[\square\]

Definition 6.1.4. Let $V$ be a value quantale. For subsets $S, T \subseteq X$ or equivalently, $S, T \in \mathcal{P}(X)$, we define the distance function $\tau: \mathcal{P}(X) \times \mathcal{P}(X) \to V$ on it by

$$\tau(S, T) = \bigwedge_{s \in S, t \in T} d(s, t)$$

and we use it to define the distance function $\zeta: \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \to V$ on collections of subsets $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{P}(X)$ by

$$\zeta(\mathcal{E}_1, \mathcal{E}_2) = \bigvee_{S \in \mathcal{E}_1, T \in \mathcal{E}_2} \tau(S, T).$$

Remark 6.1.5. Note that the Triangle Inequality for $\tau$, i.e., for every $S, T, R \in \mathcal{P}(X)$, $\tau(S, R) \leq \tau(S, T) + \tau(T, R)$ does not always hold, but there are some interesting properties of $\tau$ given in the following proposition.

Proposition 6.1.6. Let $F, G, H \in \mathcal{P}(X)$ and $x \in X$. Then the following hold.
CHAPTER 6. CAUCHY COMPLETION

1. \( \tau(F, x) + \tau(x, G) = \bigwedge_{f \in F, g \in G} [d(f, x) + d(x, g)] \).

2. \( \tau(F, G) \leq \tau(F, x) + \tau(x, G) \).

3. \( \tau(F, H) \leq \tau(F \cap G, H) \) if \( F \cap G \neq \emptyset \) and \( \tau(F, H) \leq \tau(F, G \cap H) \) if \( G \cap H \neq \emptyset \).

**Proof.**

1. By Infinite Distributive Law of addition and Lemma 2.2.21(1), we have
   \[
   \tau(F, x) + \tau(x, G) = \bigwedge_{f \in F} d(f, x) + \bigwedge_{g \in G} d(x, g)
   \]
   \[
   = \bigwedge_{f \in F} \Big[ d(f, x) + \bigwedge_{g \in G} d(x, g) \Big]
   \]
   \[
   = \bigwedge_{f \in F} \bigwedge_{g \in G} [d(f, x) + d(x, g)]
   \]
   \[
   = \bigwedge_{f \in F, g \in G} [d(f, x) + d(x, g)].
   \]

2. By the Triangle Inequality and the result in part (1), we can easily see that
   \[
   \tau(F, G) = \bigwedge_{f \in F, g \in G} d(f, g)
   \]
   \[
   \leq \bigwedge_{f \in F, g \in G} [d(f, x) + d(x, g)]
   \]
   \[
   = \bigwedge_{f \in F} d(f, x) + \bigwedge_{g \in G} d(x, g)
   \]
   \[
   = \tau(F, x) + \tau(x, G).
   \]

3. For the first inequality, we need to show that
   \[
   \tau(F, H) = \bigwedge_{f \in F, h \in H} d(f, h) \leq \bigwedge_{g \in F \cap G, h' \in H} d(g, h') = \tau(F \cap G, H)
   \]
   which already holds by Proposition 2.2.19(3), since \( F \cap G \subseteq F \). The argument for the second inequality follows similarly.

\[\square\]

**Proposition 6.1.7.** Let \( X \) be a \( V \)-space. Then for every \( x, y \in X \), \( \epsilon > 0 \) and \( \delta > 0 \) it holds that
   \[
   d(x, y) - \epsilon - \delta \leq \tau(x, y^\delta).
   \]
Proof. Let $\epsilon > 0$, $\delta > 0$ and $x, y \in X$. So we need to show that

$$d(x, y) - \epsilon - \delta \leq \bigwedge_{a \in x, b \in y} d(a, b).$$

Then by Proposition 2.2.19(1), it is equivalent to show that for every $a \in x_\epsilon$ and $b \in y_\delta$,

$$d(x, y) - \epsilon - \delta \leq d(a, b).$$

Given $a \in x_\epsilon$ and $b \in y_\delta$, we have $d(x, a) \leq \epsilon$ and $d(b, y) \leq \delta$. Then by the Triangle Inequality, we obtain

$$d(x, y) \leq d(x, a) + d(a, b) + d(b, y) \leq \epsilon + d(a, b) + \delta,$$

thus $d(x, y) - \epsilon - \delta \leq d(a, b)$. Since $a \in x_\epsilon$ and $b \in y_\delta$ are arbitrary, the result follows.

$\square$

**Definition 6.1.8.** Let $X$ be a $V$-space. Then $X$ satisfies uniformly vanishing asymmetry (or UVA) if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) \leq \delta$ implies that $d(y, x) \leq \epsilon$.

**Example 6.1.9.** Every symmetric $V$-space satisfies UVA.

**Example 6.1.10.** The set of extended real numbers $[0, \infty]$ with the distance function $d: [0, \infty] \times [0, \infty] \to [0, \infty]$ defined by

$$d(x, y) = \begin{cases} y - x, & y > x \\ 2(x - y), & y \leq x \end{cases}$$

satisfies UVA property.

**Proposition 6.1.11.** Let $X$ be a $V$-space satisfying UVA and let $\epsilon > 0$. If $\delta > 0$ corresponds to $\epsilon$ in the definition of UVA, then for $x, y \in X$ we have

$$d(x, y) - 2 \cdot \epsilon \leq \tau(x_\epsilon, y_\delta).$$

Proof. Let $\epsilon > 0$. By UVA property, there exists a $\delta > 0$ corresponding to $\epsilon > 0$. By Proposition 2.2.19(1), it suffices to show that for every $a \in x_\epsilon$ and $b \in y_\delta$,

$$d(x, y) - 2 \cdot \epsilon \leq d(a, b).$$
Given $a \in x, b \in y$, we have $d(x, a) \leq \epsilon$ and $d(y, b) \leq \delta$ which implies $d(b, y) \leq \epsilon$ by UVA. Then
\[
d(x, y) \leq d(x, a) + d(a, b) + d(b, y) \leq \epsilon + d(a, b) + \epsilon,
\]
thus $d(x, y) - 2 \cdot \epsilon \leq d(a, b)$. Since $a \in x, b \in y$ are arbitrary, then the result follows.

**Corollary 6.1.12.** Let $X$ be a $V$-space satisfying UVA. Given $\epsilon_1 > 0$ and $x, y \in X$, there exists $\epsilon_2 > 0$ such that if $\tau(x_{\delta_1}, y_{\delta_2}) \leq \epsilon_2 \forall \delta_1, \delta_2 > 0$, then $d(x, y) \leq \epsilon_1$.

**Proof.** Let $\epsilon_1 > 0$. By Proposition 2.3.18, there is a $\epsilon_2 > 0$ such that $3 \cdot \epsilon_2 \leq \epsilon_1$. Choose $\delta_1$ corresponding to $\epsilon_2$ in the definition of UVA. Then
\[
d(x, y) \leq \tau(x_{\epsilon_2}, y_{\epsilon_1}) + 2 \cdot \epsilon_2 \leq 3 \cdot \epsilon_2 \leq \epsilon_1.
\]

6.2 Filters

In this section, we introduce filter bases and filters. We define Cauchy filters, round filters and minimal Cauchy filters, and present a few basic results which will be used in Section 6.3 and Section 6.4.

**Definition 6.2.1.** Let $X$ be a set. Then a filter on $X$ is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ such that the following holds:

1. $\emptyset \notin \mathcal{F}$.
2. For every $A, B \subseteq X$ where $A \subseteq B$, if $A \in \mathcal{F}$, then $B \in \mathcal{F}$.
3. For every $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$.

A filter $\mathcal{G}$ such that $\mathcal{G} \subseteq \mathcal{F}$ is a sub-filter of $\mathcal{F}$.

**Definition 6.2.2.** Let $X$ be a set. Then a filter base is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\emptyset \notin \mathcal{B}$ and for every $A, B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

**Remark 6.2.3.** It immediately follows that a filter base $\mathcal{B}$ gives rise to a filter $\mathcal{F}$, the least filter containing $\mathcal{B}$, given explicitly by
\[
\mathcal{F} = \{ D \subseteq X \mid \exists C \in \mathcal{B}: C \subseteq D \}.
\]
From this point onwards, when we say a filter (or filter base) on a $V$-space, it would mean a filter (or filter base) on its underlying set. Also, we fix a value quantale $V$ and a $V$-space $X$.

**Definition 6.2.4.** Let $\mathcal{F}$ be a filter on $X$. Then $\mathcal{F}$ is Cauchy if for every $\epsilon > 0$, there exists a $x \in X$ such that $x_{\epsilon} \in \mathcal{F}$. Moreover, if $\mathcal{F}$ does not contain any proper Cauchy sub-filters, then $\mathcal{F}$ is called a minimal Cauchy filter.

**Definition 6.2.5.** Let $\mathcal{F}$ be a filter (or filter base) on $X$. Then $\mathcal{F}$ is round if for every $F \in \mathcal{F}$, there exists $\epsilon > 0$ such that $x_{\epsilon} \in \mathcal{F}$ implies that $x_{\epsilon} \subseteq F$, for every $x \in X$.

**Proposition 6.2.6.** Let $\mathcal{B}$ be a filter base on $X$ which generates the filter $\mathcal{F}$. Then $\mathcal{B}$ is round if $\mathcal{F}$ is round.

**Remark 6.2.7.** Trivial proof.

**Remark 6.2.8.** Note that it does not always hold that $\mathcal{F}$ is round if $\mathcal{B}$ is round. For instance, let $X = [0, \infty]^\text{op}_{\sigma^d}$ and $T = \{0\}$. We can easily see that the filter base $\mathcal{B} = \{T\}$ is round. In contrast, the generated filter $\mathcal{F} = \{S \subseteq X \mid 0 \in S\}$ is not round. We give a simple example to verify this: given $\epsilon > 0$, we have

$$0_{\epsilon} = [0, \epsilon] = \{y \in [0, \infty] \mid d_{\sigma^d}(0, y) = y - 0 \leq \epsilon\}$$

in $\mathcal{F}$ but $0_{\epsilon} \not\subseteq \{0, 1\}$.

**Proposition 6.2.9.** Let $\mathcal{F}$ be a filter on $X$. If $\mathcal{F}$ is Cauchy and round, then $\mathcal{F}$ is a minimal Cauchy filter.

**Proof.** Suppose $\mathcal{G} \subseteq \mathcal{F}$ is Cauchy. Then for every $F \in \mathcal{F}$, we will show that $F \in \mathcal{G}$. Since $\mathcal{F}$ is round, there exists $\epsilon > 0$ such that $x_{\epsilon} \in \mathcal{F}$ implies $x_{\epsilon} \subseteq F$, for every $x \in X$. Since $\mathcal{G}$ is Cauchy, there is an $x \in X$ with $x_{\epsilon} \in \mathcal{G}$. Then since $\mathcal{G} \subseteq \mathcal{F}$, $x_{\epsilon} \in \mathcal{F}$. This implies that $x_{\epsilon} \subseteq F$ and thus $F \in \mathcal{G}$. Since $F \in \mathcal{F}$ is arbitrary, the result follows.

**Proposition 6.2.10.** Let $\mathcal{B}$ and $\mathcal{D}$ be filter bases for Cauchy filters $\mathcal{F}$ and $\mathcal{G}$ on $X$, respectively. Then

$$\zeta(\mathcal{F}, \mathcal{G}) = \zeta(\mathcal{B}, \mathcal{D}).$$
Proof. It suffices to show that
\[ \zeta(\mathcal{F}, \mathcal{G}) \leq \zeta(\mathcal{B}, \mathcal{D}) \]
and
\[ \zeta(\mathcal{F}, \mathcal{G}) \geq \zeta(\mathcal{B}, \mathcal{D}). \]
For the first inequality, we need to show that
\[ \zeta(\mathcal{F}, \mathcal{G}) = \bigvee_{F \in \mathcal{F}, G \in \mathcal{G}} \tau(F, G) \leq \bigvee_{B \in \mathcal{B}, D \in \mathcal{D}} \tau(B, D) = \zeta(\mathcal{B}, \mathcal{D}), \]
or equivalently, for every \( F \in \mathcal{F} \) and \( G \in \mathcal{G} \),
\[ \tau(F, G) \leq \bigvee_{B \in \mathcal{B}, D \in \mathcal{D}} \tau(B, D) \]
by Proposition 2.2.19(2). Fix \( F \in \mathcal{F} \) and \( G \in \mathcal{G} \). Then there exists \( U \in \mathcal{B} \) and \( V \in \mathcal{D} \) such that \( U \subseteq F \) and \( V \subseteq G \). Then it follows from Proposition 6.1.6(3) that
\[ \tau(F, G) \leq \tau(U, V). \]
Then we obtain
\[ \tau(F, G) \leq \tau(U, V) \leq \bigvee_{B \in \mathcal{B}, D \in \mathcal{D}} \tau(B, D) = \zeta(\mathcal{B}, \mathcal{D}). \]
Since \( F \in \mathcal{F} \) and \( G \in \mathcal{G} \) are arbitrary,
\[ \zeta(\mathcal{F}, \mathcal{G}) \leq \zeta(\mathcal{B}, \mathcal{D}) \]
Now, for the second inequality, we need to show that
\[ \zeta(\mathcal{F}, \mathcal{G}) = \bigvee_{F \in \mathcal{F}, G \in \mathcal{G}} \tau(F, G) \geq \bigvee_{B \in \mathcal{B}, D \in \mathcal{D}} \tau(B, D) = \zeta(\mathcal{B}, \mathcal{D}) \]
which indeed holds by Proposition 2.2.19(3) since \( \mathcal{B} \subseteq \mathcal{F} \) and \( \mathcal{D} \subseteq \mathcal{G} \).

6.3 Cauchy Completeness

In this section of the chapter, we define what it means for a \( V \)-space to be Cauchy complete and op-Cauchy complete. We also show that \( V_{st} \) and \( V_{st}^{op} \) are both Cauchy complete.
Definition 6.3.1. Let $X$ be a $V$-space. A filter $\mathcal{F}$ on $X$ converges to $x \in X$ (or simply $\mathcal{F} \to x$) if for every $\epsilon > 0$, the ball $x_{\epsilon} \in \mathcal{F}$. Then $X$ is said to be Cauchy complete if every Cauchy filter on $X$ converges. Also, $X$ is said to be op-Cauchy complete if $X^{op}$ is Cauchy complete.

Lemma 6.3.2. For a value quantale $V$, the $V$-space $V_{st}$ is Cauchy complete and op-Cauchy complete.

Proof. We first show that $V_{st}$ is Cauchy complete. Let $\mathcal{F}$ be a Cauchy filter in $V_{st}$. By definition of a Cauchy filter, for every $\epsilon > 0$, there exists a $v(\epsilon) \in V$ ($v(\epsilon)$ means that the choice of $v$ is dependent on the value of $\epsilon$) such that $v(\epsilon)_{\epsilon} \in \mathcal{F}$ and we denote

$$x = \bigwedge_{\epsilon > 0} v(\epsilon).$$

Now we go on to show that $\mathcal{F}$ converges to $x$. Let $\delta > 0$, so

$$v(\delta)_{\delta} = \{y \in V \mid d_{st}(v(\delta), y) \leq \delta\} \in \mathcal{F}.$$

We first verify that $v(\delta)_{\delta} \subseteq x_{\delta}$. Let $y \in v(\delta)_{\delta}$. Then

$$d_{st}(v(\delta), y) = v(\delta) - y \leq \delta$$

which is equivalent to $v(\delta) \leq \delta + y$. Since $x \leq v(\delta)$, we have $x \leq v(\delta) \leq \delta + y$ which implies that

$$d(x, y) = x - y \leq \delta,$$

thus $y \in x_{\delta}$. Since $y \in v(\delta)_{\delta}$ is arbitrary, $v(\delta)_{\delta} \subseteq x_{\delta}$. Then since $v(\delta)_{\delta} \in \mathcal{F}$, we obtain $x_{\delta} \in \mathcal{F}$. Since $\delta > 0$ is arbitrary, $\mathcal{F}$ converges to $x$.

Next we show that $V_{st}$ is op-Cauchy complete, or equivalently, $V_{st}^{op}$ is Cauchy complete. Let $\mathcal{G}$ be a Cauchy filter on $V_{st}^{op}$. For every $\epsilon > 0$, we have $v(\epsilon) \in V$ such that $v(\epsilon)_{\epsilon} \in \mathcal{G}$ and denote

$$x = \bigvee_{\epsilon > 0} v(\epsilon).$$

It is enough to show that $\mathcal{G}$ converges to $x$. Let $\delta > 0$, so we have

$$v(\delta)_{\delta} = \{y \in V \mid d_{st}^{op}(v(\delta), y) \leq \delta\} \in \mathcal{G}.$$

We first verify that $v(\delta)_{\delta} \subseteq x_{\delta}$. Let $y \in v(\delta)_{\delta}$. Then

$$d_{st}^{op}(v(\delta), y) = d(y, v(\delta)) = y - v(\delta) \leq \delta.$$
Since $v(\delta) \leq x$, we have
\[ y \leq v(\delta) + \delta \leq x + \delta \]
which means
\[ d^{\text{op}}(x, y) = y - x \leq \delta, \]
thus $y \in x_\delta$. Since $y \in v(\delta)_\delta$ is arbitrary, $v(\delta)_\delta \subseteq x_\delta$. Since $v(\delta)_\delta \in \mathcal{G}$, the $x_\delta \in \mathcal{G}$. Then since $\delta > 0$ is arbitrary, $\mathcal{G}$ converges to $x$, therefore $V_{st}^{\text{op}}$ is Cauchy complete.

Remark 6.3.3. From this result, we conclude that $V_{st}$ and $V_{st}^{\text{op}}$ are both Cauchy and op-Cauchy complete.

### 6.4 The Completion Construction

In this section, we present a new completion construction of $V$-spaces satisfying UVA. This construction takes motivation from the standard completion construction of uniform spaces. The situation changes with quasi-metric spaces which do not satisfy symmetry. Many different completion constructions exists such as half-completion, bicompletion [1], Smyth completion [12] and so on, since different concepts of completeness arises.

Consider the sets
\[ U_\epsilon = \{ (x, y) \in X \times X \mid d(x, y) < \epsilon \} \]
and
\[ U^{\text{op}}_\epsilon = \{ (x, y) \in X \times X \mid d(y, x) < \epsilon \} \]
in a metric space $(X, d)$. We have $U_\epsilon = U^{\text{op}}_\epsilon$ due to symmetry thus we have a functor
\[
\text{Met} \xrightarrow{U_\epsilon} \text{Unif}.
\]
But this is not the case for quasi-metric spaces. When symmetry is not satisfied, it need not hold that $U_\epsilon = U^{\text{op}}_\epsilon$. So we have two functors
\[
\text{QMet} \xrightarrow{U_\epsilon} \text{QUnif}.
\]
Then the equalizer is easily seen to be the category of quasi-metric spaces satisfying UVA, denoted $\text{QMet}_{\text{uva}}$, together with the inclusion functor
\[
\text{QMet}_{\text{uva}} \rightarrow \text{QMet},
\]
thus obtaining

\[ \text{QMet}^{\text{sea}} \longrightarrow \text{QMet} \xrightarrow{U(-)} \text{QUnif}. \]

Since a metric space is an example of a separated symmetric \( V \)-space, the diagram can be generalized to

\[ \text{SMet}^{\text{sea}} \longrightarrow \text{SMet} \xrightarrow{U(-)} \text{QUnif} \]

where \( \text{SMet}^{\text{sea}} \) is the category of separated \( V \)-spaces satisfying UVA. Note that these categories and functors are respectively objects and arrows in \( \text{Cat} \).

In the note by Weiss [58] based on results in [15], it is shown that the category of topological spaces is equivalent to a category consisting of all \( V \)-spaces, where \( V \) varies over all value quantales. This implies that sequences are inadequate for the completion construction of \( V \)-spaces and thus filters are utilized.

**Proposition 6.4.1.** Let \( \mathcal{F} \) be a filter on \( X \). Then the collections

\[ B_{>0} = \{ F \in \mathcal{F}, \epsilon > 0 \} \]

and

\[ B_{>0}^c = \{ F^c \mid F \in \mathcal{F}, \epsilon > 0 \} \]

are filter bases.

**Proof.** Note that \( B_{>0} \) is non-empty since \( X \in \mathcal{F} \) and also \( \emptyset \) is not in the collection since \( \emptyset \notin \mathcal{F} \). Next, we show that for \( F, F' \in \mathcal{F} \),

\[ (F \cap F') \epsilon \epsilon' \subseteq F \epsilon \cap F' \epsilon'. \]

Let \( y \in (F \cap F') \epsilon \epsilon' \) and \( \epsilon, \epsilon' > 0 \). This means that there exists an \( x \in F \cap F' \) such that \( d(x, y) \leq \epsilon \wedge \epsilon' \) then simply \( d(x, y) \leq \epsilon \) and \( d(x, y) \leq \epsilon' \). By intersection, we see that \( x \in F \) and \( x \in F' \). Since \( d(x, y) \leq \epsilon \), \( y \in F \epsilon \) and since \( d(x, y) \leq \epsilon' \), \( y \in F' \epsilon' \), thus \( y \in F \epsilon \cap F' \epsilon' \). The argument for \( B_{>0}^c \) follows similarly.

\[ \square \]

**Remark 6.4.2.** Using the filter base \( B_{>0} \), we generate the filter \( \mathcal{F}_{>0} \). Similarly, using \( B_{>0}^c \), we obtain \( \mathcal{F}_{>0}^c \). Remark 6.2.3 explains more of the construction of filters using filter bases.

**Proposition 6.4.3.** Let \( \mathcal{F} \) be a filter on \( X \). If \( \mathcal{F} \) is Cauchy, then \( \mathcal{F}_{>0} \) is Cauchy.
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\textbf{Proof.} Let $\epsilon > 0$. Then by Lemma 2.3.19, there is a $\delta > 0$ such that $\delta + \delta \leq \epsilon$. Since $\mathcal{F}$ is Cauchy, there is an $x \in X$ such that $x_\delta \in \mathcal{F}$. Since $(x_\delta)_\delta \in \mathcal{F}_\succ$ and

$$(x_\delta)_\delta \subseteq x_{\delta + \delta} \subseteq x_\epsilon$$

by Proposition 6.1.3(1) and (2), it follows that $x_\epsilon \in \mathcal{F}_\succ$. \hfill \Box

\textbf{Proposition 6.4.4.} Let $X$ be a $V$-space satisfying UVA. If $\mathcal{F}$ is a filter, then $\mathcal{F}_\succ$ is a round filter.

\textbf{Proof.} It suffices to show that for a given $\epsilon > 0$ and $F_\epsilon \in \mathcal{F}_\succ$, there exists $\delta > 0$ such that if $y_\delta \in \mathcal{F}_\succ$, then $y_\delta \subseteq F_\epsilon$ for every $y \in X$. Let $\epsilon > 0$. Then by Proposition 2.3.19, there is $\delta_1 > 0$ such that $\delta_1 + \delta_1 \leq \epsilon$ and let $\delta_2 > 0$ correspond to $\delta_1$ which is guaranteed by UVA property. We have $\delta_1 \wedge \delta_2 > 0$ since $V$ is a value distributive lattice, so set $\delta = \delta_1 \wedge \delta_2$. Suppose that $y_\delta \in \mathcal{F}_\succ$ for $y \in X$. Then

$$F'_\epsilon \subseteq F'_\epsilon \subseteq y_\delta$$

for some $F' \in \mathcal{F}$ since $\mathcal{F}_\succ$ is generated by filter base $B_\succ$. Let $x \in F \cap F'$ where $F \in \mathcal{F}$. Then $x \in y_\delta$ and so $d(y, x) \leq \delta$. If $z \in y_\delta$, then $d(y, z) \leq \delta$, and so

$$d(x, z) \leq d(x, y) + d(y, z) \leq \delta_1 + \delta_1 \leq \epsilon,$$

then $z \in x_\epsilon$ which implies that $z \in F_\epsilon$. Since $z \in y_\delta$ is arbitrary, it follows that $y_\delta \subseteq F_\epsilon$. \hfill \Box

\textbf{Corollary 6.4.5.} Let $X$ be a $V$-space satisfying UVA and $\mathcal{F}$ a Cauchy filter on $X$. Then $\mathcal{F}_\succ$ is a minimal Cauchy filter.

\textbf{Proof.} By Propositions 6.4.3 and 6.4.4, it follows that $\mathcal{F}_\succ$ is Cauchy and round, thus by Proposition 6.2.9, it is minimal. \hfill \Box

\textbf{Corollary 6.4.6.} Let $X$ be a $V$-space satisfying UVA and $\mathcal{F}$ be a filter on $X$. Then the filter $\mathcal{F}_\succ$ is minimal Cauchy, if and only if, $\mathcal{F}$ is Cauchy and round.

\textbf{Proof.} Corollary 6.4.5 above proves one direction. For the other direction, if $\mathcal{F}_\succ$ is a minimal Cauchy filter, then $\mathcal{F} = \mathcal{F}_\succ$, thus $\mathcal{F}$ is Cauchy and also round by Proposition 6.4.4. \hfill \Box
Proposition 6.4.7. Let $X$ be a $V$-space. Then the sets

$$B_x = \{ x_\epsilon \mid \epsilon \succ 0 \}$$

and

$$B_x^* = \{ x_\epsilon^* \mid \epsilon \succ 0 \}$$

are both filter bases.

Proof. We first show that $B_x$ is a filter base. We know that $B_x$ is non-empty since $X \in B_x$ and also $\emptyset \not\in B_x$ since for every $\epsilon \succ 0$, at least $x \in x_\epsilon$. Lastly, given $x_\epsilon, x_{\epsilon'} \in B_x$ for some $\epsilon \succ 0$ and $\epsilon' \succ 0$. Since $V$ is a value distributive lattice, we have $\epsilon \land \epsilon' \succ 0$. So we only have to show that $x_{\epsilon \land \epsilon'} \subseteq x_\epsilon$ and $x_{\epsilon \land \epsilon'} \subseteq x_{\epsilon'}$. Let $y \in x_{\epsilon \land \epsilon'}$. This means

$$d(x, y) \leq \epsilon \land \epsilon' \leq \epsilon$$

and

$$d(x, y) \leq \epsilon \land \epsilon' \leq \epsilon',$$

thus $y \in x_\epsilon$ and $y \in x_{\epsilon'}$. Since $y \in x_{\epsilon \land \epsilon'}$ is arbitrary, it follows that $x_{\epsilon \land \epsilon'} \subseteq x_\epsilon$ and $x_{\epsilon \land \epsilon'} \subseteq x_{\epsilon'}$.

Since $B_x^*$ is a dual construction on $X^{op}$, by Theorem 2.1.22, it is a filter base.

Remark 6.4.8. Given $V$-space $X$ and $x \in X$, let $\mathcal{F}_x$ be the filter generated by the filter base $B_x$ and similarly for $\mathcal{F}_x^*$ by $B_x^*$. It is easy enough to verify that $\mathcal{F}_x$ and $\mathcal{F}_x^*$ are Cauchy. Also, note that the property $\mathcal{F}_x = \mathcal{F}_x^*$ does not usually hold due to asymmetry of $X$.

Below, we present a result which shows that UVA is sufficient for the equality $\mathcal{F}_x = \mathcal{F}_x^*$.

Proposition 6.4.9. Let $X$ be a $V$-space satisfying UVA. Then $\mathcal{F}_x = \mathcal{F}_x^*$ holds for every $x \in X$.

Proof. Suppose $X$ satisfies UVA. It suffices to show that $B_x^* = B_x$, that is, $B_x \subseteq B_x^*$ and $B_x^* \subseteq B_x$ for a fixed $x \in X$. For the first inclusion, let $x_\epsilon \in B_x$ for some $\epsilon \succ 0$. By the UVA property, there exists a $\delta \succ 0$ such that for every $y \in X$, $d(y, x) \leq \delta$ implies $d(x, y) \leq \epsilon$. Let $z \in x^\delta$. So $d(z, x) \leq \delta$, then it follows by UVA that $d(x, z) \leq \epsilon$, thus $z \in x_\epsilon$. Since $z \in x^\delta$ is arbitrary, it follows that $x^\delta \subseteq x_\epsilon$ and since $x^\delta \in B_x$, we obtain

$$x_\epsilon \in B_x^*.$$
Since \( \epsilon > 0 \) is arbitrary, it follows that \( B_x \subseteq B^x \). The argument for \( B^x \subseteq B_x \) follows similarly. \( \square \)

**Remark 6.4.10.** The bifurcations in the theories of completions of quasi metric spaces are related to the fact that in general \( \mathcal{F}_x \neq \mathcal{F}^x \), since the inclusion into the completion is typically given by the assignment \( x \mapsto \mathcal{F}_x \).

**Proposition 6.4.11.** Let \( X \) be a \( V \)-space satisfying UVA and \( x \in X \). Then \( \mathcal{F}_x \) is a minimal Cauchy filter.

**Proof.** We can easily see that \( \mathcal{F}_x \) is Cauchy since it is generated by

\[
B_x = \{ x_\epsilon \mid \epsilon > 0 \},
\]

so it only remains to show that \( \mathcal{F}_x \) is round by Proposition 6.2.9. Let \( x \in X \) and \( F \in \mathcal{F}_x \). Then by definition of \( \mathcal{F}_x \), there is an \( \epsilon > 0 \) such that \( x_\epsilon \in \mathcal{F}_x \) and \( x_\epsilon \subseteq F \). So it suffices to show that there exists a \( \delta > 0 \) such that for every \( y \in X \),

\[
y_\delta \in \mathcal{F}_x \text{ implies that } y_\delta \subseteq x_\epsilon \subseteq F.
\]

Since \( \epsilon > 0 \), then by Lemma 2.3.19, there exists \( \delta_1 > 0 \) such that \( \delta_1 + \delta_1 \leq \epsilon \) and by UVA, we have a corresponding \( \delta_2 > 0 \) for \( \delta_1 \). Set

\[
\delta = \delta_1 \wedge \delta_2.
\]

We know \( \delta > 0 \) since \( V \) is a value distributive lattice. Suppose \( y_\delta \in \mathcal{F}_x \) for \( y \in X \). Then by definition of the filter \( \mathcal{F}_x \), we have \( x \in y_\delta \) so \( d(y, x) \leq \delta \leq \delta_2 \) implying

\[
d(x, y) \leq \delta_1.
\]

Now, let \( z \in y_\delta \) which means \( d(y, z) \leq \delta \). Then

\[
d(x, z) \leq d(x, y) + d(y, z) \leq \delta_1 + \delta \leq \delta_1 + \delta_1 \leq \epsilon,
\]

so \( z \in x_\epsilon \). Since \( z \in y_\delta \) is arbitrary, \( y_\delta \subseteq x_\epsilon \subseteq F \) which completes the proof. \( \square \)

**Definition 6.4.12.** Let \( X \) be a \( V \)-space satisfying UVA. We define \( \tilde{X} \subseteq \mathcal{P}(\mathcal{P}(X)) \) to be the set of all Cauchy filters on \( X \) and \( \hat{X} \subseteq \tilde{X} \) to be the set of all minimal Cauchy filters.
Lemma 6.4.13. Let $X$ be a $V$-space satisfying UVA. Then $\tilde{X} = (\tilde{X}, \zeta)$ is a $V$-space, which itself satisfies UVA.

Proof. We show that $\tilde{X}$ satisfies (1) Reflexivity, (2) Triangle Inequality and (3) UVA.

1. Reflexivity: For every filter $\mathcal{F} \in \tilde{X}$, since it holds that $F \cap G \neq \emptyset$ for every $F, G \in \mathcal{F}$,

$$\tau(F, G) = \bigwedge_{f \in F, g \in G} d(f, g) = 0.$$ 

Thus we obtain

$$\zeta(\mathcal{F}, \mathcal{F}) = \bigvee_{F, G \in \mathcal{F}} \tau(F, G) = \bigvee_{F, G \in \mathcal{F}} 0 = 0.$$

2. Triangle Inequality: To establish that

$$\zeta(\mathcal{F}, \mathcal{H}) \leq \zeta(\mathcal{F}, \mathcal{G}) + \zeta(\mathcal{G}, \mathcal{H}),$$

it suffices to show, for fixed $\epsilon \succ 0$, $F \in \mathcal{F}$ and $H \in \mathcal{H}$, that there exists $G \in \mathcal{G}$ such that

$$\tau(F, H) \leq \tau(F, G) + \tau(G, H) + \epsilon.$$

By Lemma 2.3.19, there exists a $\delta \succ 0$ such that $\delta + \delta \leq \epsilon$ and by UVA, there exists a $\delta' \succ 0$ corresponding to $\delta$. Set $\eta = \delta \land \delta'$ where $\eta \succ 0$ since $V$ is a value distributive lattice. As $\mathcal{G}$ is Cauchy, there is an $x \in X$ such that $x_\eta \in \mathcal{G}$. Denote $G = x_\eta$. Then

$$\tau(F, G) + \tau(G, H) + \epsilon = \left[ \bigwedge_{f \in F, y \in G} d(f, y) \right] + \left[ \bigwedge_{z \in G, h \in H} d(z, h) \right] + \epsilon$$

$$\geq \left[ \bigwedge_{f \in F, y, z \in G, h \in H} d(f, y) + d(z, h) \right] + \delta + \delta$$

$$= \bigwedge_{f \in F, y, z \in G, h \in H} [d(f, y) + \delta + \delta + d(z, h)]$$

$$\geq \bigwedge_{f \in F, y, z \in G, h \in H} [d(f, y) + d(y, x) + d(x, z) + d(z, h)]$$

$$\geq \bigwedge_{f \in F, h \in H} [d(f, x) + d(x, h)]$$

$$= \tau(F, x) + \tau(x, H)$$

$$\geq \tau(F, H).$$
Since
\[ \tau(F, G) \leq \bigvee_{F' \in \mathcal{F}, G' \in \mathcal{G}} \tau(F', G') \]
and
\[ \tau(G, H) \leq \bigvee_{G' \in \mathcal{G}, H' \in \mathcal{H}} \tau(G', H') \]
we obtain
\[ \tau(F, H) \leq \tau(F, G) + \tau(G, H) + \epsilon \]
\[ \leq \bigvee_{F' \in \mathcal{F}, G' \in \mathcal{G}} \tau(F', G') + \bigvee_{G' \in \mathcal{G}, H' \in \mathcal{H}} \tau(G', H') + \epsilon \]
\[ = \zeta(\mathcal{F}, \mathcal{G}) + \zeta(\mathcal{G}, \mathcal{H}) + \epsilon. \]

Then since \( \epsilon \succ 0 \), \( F \in \mathcal{F} \) and \( H \in \mathcal{H} \) are arbitrary, the result follows.

3. Next, we show that \( \hat{X} \) satisfies the UVA property. Let \( \epsilon \succ 0 \). Then by Theorem 2.3.18, there exists an \( \epsilon' \succ 0 \) such that \( 3 \cdot \epsilon' \leq \epsilon \). Since \( X \) is UVA, there exists a \( \delta' \succ 0 \) corresponding to \( \epsilon' \succ 0 \).

Let \( \mathcal{F}, \mathcal{G} \in \hat{X} \). For the \( \delta' \succ 0 \), by the Corollary 6.1.12, there exists a corresponding \( \delta \succ 0 \). Suppose
\[ \zeta(\mathcal{F}, \mathcal{G}) = \bigvee_{F \in \mathcal{F}, G \in \mathcal{G}} \tau(F, G) \leq \delta, \]
then by Proposition 2.2.19(2), \( \tau(F, G) \leq \delta \) for every \( F \in \mathcal{F} \) and \( G \in \mathcal{G} \). We know that \( \mathcal{F} \) and \( \mathcal{G} \) are Cauchy filters, so there exists \( x, y \in X \) such that \( x_{\epsilon'} \in \mathcal{F} \) and \( y_{\delta'} \in \mathcal{G} \). We need to show that
\[ \zeta(\mathcal{G}, \mathcal{F}) = \bigvee_{F \in \mathcal{F}, G \in \mathcal{G}} \tau(G, F) \leq \epsilon, \]
which is equivalent to \( \tau(G, F) \leq \epsilon \) for every \( G \in \mathcal{G} \) and \( F \in \mathcal{F} \) by Proposition 2.2.19(2). Let \( F_0 \in \mathcal{F} \) and \( G_0 \in \mathcal{G} \). Then it suffices to show that
\[ \tau(G_0, F_0) = \bigwedge_{f \in F_0, g \in G_0} d(g, f) \leq \epsilon. \]

We know that \( F_0 \cap x_{\epsilon'} \in \mathcal{F} \) and \( G_0 \cap y_{\delta'} \in \mathcal{G} \). Then for \( b \in G_0 \cap y_{\delta'} \) and \( a \in F_0 \cap x_{\epsilon'} \), it holds that
\[ d(b, a) \leq d(b, y) + d(y, x) + d(x, a). \]
Since \( a \in x_{\epsilon'}, d(x, a) \leq \epsilon' \) and since \( b \in y_{\delta'} \), we have \( d(y, b) \leq \delta' \), which by UVA property implies that \( d(b, y) \leq \epsilon' \). Also, since \( \tau(x_{\epsilon'}, y_{\delta'}) \leq \delta \), then \( d(x, y) \leq \delta' \) by Corollary 6.1.12, which in turn implies that \( d(y, x) \leq \epsilon' \) by UVA property. So we obtain
\[
d(b, a) \leq d(b, y) + d(y, x) + d(x, a) \leq \epsilon' + \epsilon' + \epsilon' \leq \epsilon,
\]
thus
\[
\tau(G_0, F_0) = \bigwedge_{f \in F_0, g \in G_0} d(g, f) \leq d(b, a) \leq \epsilon.
\]
Since \( F_0 \in \mathcal{F} \) and \( G_0 \in \mathcal{G} \) are arbitrary, it follows that \( \zeta(\mathcal{G}, \mathcal{F}) \leq \epsilon \).

Proposition 6.4.14. Let \( X \) be a \( V \)-space. For \( x, y \in X \), we define a relation \( x \sim y \) whenever
\[
d(x, y) = d(y, x) = 0.
\]
Then \( \sim \) is an equivalence relation.

Proof. Trivial.

Remark 6.4.15. Through the equivalence relation \( \sim \) on \( X \), we obtain the set of all equivalence classes, denoted by \( X_0 \). The pair \( X_0 = (X_0, d_0) \), where
\[
d_0([x], [y]) = d(x, y)
\]
(which is clearly well-defined due to the Triangle Inequality) is a separated \( V \)-space. Note that \( (\tilde{X})_0 \) from here onwards will simply be denoted by \( \tilde{X}_0 \). For the inclusion functor
\[
\text{SMet}_V \longrightarrow \text{Met}_V,
\]
we have a left adjoint
\[
-\circ: \text{Met}_V \longrightarrow \text{SMet}_V
\]
for which the object function is given by the construction above.

Proposition 6.4.16. Let \( X \) be a \( V \)-space satisfying UVA. Then \( \tilde{X} = (\tilde{X}, \zeta) \) is a separated \( V \)-space.
Proof. The proof is given in three parts. We first verify that

\[ F \cap G = \{ Y \subseteq X \mid Y \in F \text{ and } Y \in G \} \]

is a filter given Cauchy filters \( F \) and \( G \). For the second part, we show that if \( \zeta(F, G) = \zeta(G, F) = 0 \) for \( F, G \in \tilde{X} \), then \( F \cap G \) is Cauchy. For the last part, we prove that if \( F, G \in \hat{X} \) and \( \zeta(F, G) = \zeta(G, F) = 0 \), then

\[ F = F \cap G = G. \]

1. We first verify that for Cauchy filters \( F \) and \( G \), \( F \cap G \) is a filter but not necessarily Cauchy.

   (a) \( F \cap G \) is non-empty: We know that \( X \in F \) and \( X \in G \), then \( X \in F \cap G \).
   (b) \( \emptyset \notin F \cap G \): Since \( \emptyset \notin F \) and \( \emptyset \notin G \), then \( \emptyset \notin F \cap G \).
   (c) If \( F \in F \cap G \) and \( F \subseteq F' \), then \( F' \in F \cap G \): Suppose \( F \in F \cap G \) and \( F \subseteq F' \). Since \( F \in F \cap G \), \( F \in F \) and \( F \in G \). Since \( F \) and \( G \) are filters, \( F \subseteq F' \) implies that \( F' \in F \) and \( F' \in G \), thus

\[ F' \in F \cap G. \]

   (d) If \( F, F' \in F \cap G \), then \( F \cap F' \in F \cap G \): Suppose \( F, F' \in F \cap G \). This means \( F, F' \in F \) and \( F, F' \in G \). Since \( F \) and \( G \) are filters, then \( F \cap F' \in F \) and \( F \cap F' \in G \), thus

\[ F \cap F' \in F \cap G. \]

2. Next, we show that for Cauchy filters \( F, G \), if \( \zeta(F, G) = 0 = \zeta(G, F) \), then \( F \cap G \) is a Cauchy filter. Let \( \epsilon > 0 \). By Proposition 2.3.18, there exists a \( \delta_1 > 0 \) such that \( 3 \cdot \delta_1 \leq \epsilon \). By UVA, there exists a \( \delta_2 > 0 \) corresponding to \( \delta_1 \). Set \( \delta = \delta_1 \wedge \delta_2 \). Since \( F \) is Cauchy, there exists \( x \in X \) such that \( x_{\delta_1} \in F \). This means that for all \( z \in x_{\delta_1}, d(x, z) \leq \delta_1 \). Since \( G \) is Cauchy, there exists \( y \in X \) such that \( y_{\delta_2} \in G \). This means that for all \( a \in y_{\delta_2}, d(y, a) \leq \delta \). Let \( a \in y_{\delta_2} \). Since \( X \) satisfies UVA, we know that

\[ d(x, y) \leq \tau(x_{\delta_1}, y_{\delta_2}) + 2\delta_1. \]

Since \( \zeta(F, G) = \bigvee_{F \in F, G \in G} \tau(F, G) = 0 \), then by Proposition 2.2.19(2),

\[ \tau(F, G) = 0 \]

for every \( F \in F \) and \( G \in G \). This means \( \tau(x_{\delta_1}, y_{\delta_2}) = 0 \), thus \( d(x, y) \leq 2 \cdot \delta_1. \)
Then by Triangle Inequality, we obtain
\[ d(x, a) \leq d(x, y) + d(y, a) \leq 2 \cdot \delta_2 + \delta \leq 3 \cdot \delta_1 \leq \epsilon, \]
thus \( a \in x_\epsilon \). Since \( a \in y_\delta \) is arbitrary, \( y_\delta \subseteq x_\epsilon \) which implies \( x_\epsilon \in \mathcal{G} \). By Proposition 6.1.3(1), we know \( x_{\delta_1} \subseteq x_\epsilon \), so \( x_\epsilon \in \mathcal{F} \). Then it follows that \( x_\epsilon \in \mathcal{F} \cap \mathcal{G} \), thus since \( \epsilon > 0 \) is arbitrary, \( \mathcal{F} \cap \mathcal{G} \) is Cauchy.

3. For the last part, suppose
\[ \zeta(\mathcal{F}, \mathcal{G}) = 0 = \zeta(\mathcal{G}, \mathcal{F}) \]
where \( \mathcal{F}, \mathcal{G} \in \hat{X} \). Then by part (1) and (2) above, it holds that \( \mathcal{F} \cap \mathcal{G} \) is a Cauchy filter. We know \( \mathcal{F} \cap \mathcal{G} \subseteq \mathcal{F} \) and \( \mathcal{F} \cap \mathcal{G} \subseteq \mathcal{G} \). Then since \( \mathcal{F} \) and \( \mathcal{G} \) are minimal Cauchy filters, we obtain
\[ \mathcal{F} = \mathcal{F} \cap \mathcal{G} = \mathcal{G}, \]
thus \( \hat{X} \) is separated.

\[ \square \]

**Theorem 6.4.17.** Let \( X \) be a \( V \)-space satisfying UVA. Then \( \hat{X} \) is Cauchy complete.

**Proof.** We need to show that every Cauchy filter on \( \hat{X} \) converges to a minimal Cauchy filter on \( X \). This means given a Cauchy filter \( \mathcal{A} \) on \( \hat{X} \) and \( \epsilon > 0 \), there exists a minimal Cauchy filter \( \mathcal{L} \in \hat{X} \) such that
\[ \mathcal{L}_\epsilon = \{ \mathcal{G} \in \hat{X} \mid \zeta(\mathcal{L}, \mathcal{G}) \leq \epsilon \} \in \mathcal{A}. \]
The proof is given below in three parts.

1. We construct a filter \( \mathcal{F} \) as follows: \( F \in \mathcal{F} \) exactly when there is an \( A \in \mathcal{A} \) such that \( F \in \bigcap A \), or equivalently,
\[ F \in \bigcap_{\mathcal{G} \in A} \mathcal{G}. \]
We first verify that \( \mathcal{F} \) is a filter.

(a) \( \mathcal{F} \) is non-empty: \( X \in \mathcal{F} \) since \( X \in \bigcap A \) for every subset of Cauchy filters \( A \in \mathcal{A} \).
(b) $\emptyset \notin \mathcal{F}$: Since $\emptyset \notin \mathcal{G}$ for every $\mathcal{G} \in A$ for any $A \in A$, then $\emptyset \notin \bigcap A$ which means $\emptyset \notin \mathcal{F}$.

(c) If $F, G \in \mathcal{F}$ and $F \subseteq G$, then $G \in \mathcal{F}$: Suppose $F \in \mathcal{F}$ and $F \subseteq G$. This means there exists an $A \in A$ such that $F \in \bigcap A$ or equivalently, $F \in \mathcal{G}$ for every filter $\mathcal{G} \in A$. Since $F \subseteq G$, we have $G \in \mathcal{G}$ for every $\mathcal{G} \in A$ which implies that

$$G \in \bigcap A,$$

thus $G \in \mathcal{F}$.

(d) if $F, G \in \mathcal{F}$, then $F \cap G \in \mathcal{F}$: Suppose $F, G \in \mathcal{F}$. This means $\exists A, B \in A$ such that $F \in \bigcap A$ and $G \in \bigcap B$ or equivalently, $F \in \mathcal{G}$ for every filter $\mathcal{A} \in A$ and $G \in \mathcal{B}$ for every $\mathcal{B} \in B$. Since $A, B \in A$, $A \cap B \in A$ and $A \cap B \neq \emptyset$ so for every filter

$$\mathcal{K} \in A \cap B,$$

we have $F \in \mathcal{K}$ and $G \in \mathcal{K}$, thus $F \cap G \in \mathcal{K}$ for every $\mathcal{K} \in A \cap B$. This implies

$$F \cap G \in A \cap B$$

which means $F \cap G \in \mathcal{F}$.

2. Next, we show that $\mathcal{F}$ is Cauchy, i.e., $\mathcal{F} \in \breve{X}$. Let $\epsilon > 0$. We need to show that there exists a $x \in X$ such that $x_\epsilon \in \mathcal{F}$. This is equivalent to showing that there exist an $x \in X$ and $A \in A$ such that $x_\epsilon \in \bigcap A$. By Proposition 2.3.18, there is a $\delta_1 > 0$ such that $4 \cdot \delta_1 \leq \epsilon$. Since $A$ is Cauchy, there is a minimal Cauchy filter $\mathcal{L}$ on $X$ such that $\mathcal{L}_{\delta_1} \in A$. This means for every $\mathcal{G} \in \mathcal{L}_{\delta_1}$,

$$\zeta(\mathcal{L}, \mathcal{G}) = \bigvee_{L \in \mathcal{L}, G \in \mathcal{G}} \tau(L, G) \leq \delta_1$$

or equivalently, $\tau(L, G) \leq \delta_1$ for every $L \in \mathcal{L}$ and $G \in \mathcal{G}$. Since $\mathcal{L}$ is a Cauchy filter, there exists a $x \in X$ such that $x_{\delta_1} \in \mathcal{L}$. By UVA property, there is a $\delta_2 > 0$ corresponding to $\delta_1$. Set

$$\delta_3 = \delta_1 \wedge \delta_2$$

where $\delta_1 \wedge \delta_2 > 0$. Let $\mathcal{G} \in \mathcal{L}_{\delta_3}$. Since $\mathcal{G}$ is Cauchy, there exists a $y \in X$ such
that $y_{\delta_3} \in \mathcal{G}$. Let $a \in y_{\delta_3}$. Since $X$ is UVA, by Proposition 6.1.11, it holds that
\[
\begin{align*}
d(x, a) & \leq d(x, y) + d(y, a) \\
& \leq \tau(x_{\delta_1}, y_{\delta_3}) + 2 \cdot \delta_1 + d(y, a) \\
& \leq 3 \cdot \delta_1 + \delta_3 \leq 4 \cdot \delta_1 \leq \epsilon,
\end{align*}
\]
which means $a \in x_\epsilon$. Since $a \in y_{\delta_3}$ is arbitrary, $y_{\delta_3} \subseteq x_\epsilon$ which implies that $x_\epsilon \in \mathcal{G}$. Since the Cauchy filter $\mathcal{G} \in \mathcal{L}_\epsilon$ is arbitrary, then $x_\epsilon$ is in every Cauchy filter in $\mathcal{L}_\epsilon$, which implies $x_\epsilon \in \bigcap \mathcal{L}_\epsilon$, thus $x_\epsilon \in \mathcal{F}$.

3. For the last part, we show that $\mathcal{A}$ converges to the minimal Cauchy filter $\mathcal{F}_\epsilon$. Firstly, note that since $X$ is UVA and $\mathcal{F}$ is a Cauchy filter, by Theorem 6.4.5, $\mathcal{F}_\epsilon$ is a minimal Cauchy filter, thus $\mathcal{F}_\epsilon \in \hat{X}$.

Let $\epsilon \succ 0$. By Lemma 2.3.19, there exists a $\delta_1 \succ 0$ such that $\delta_1 + \delta_1 \leq \epsilon$. By UVA property of $\hat{X}$, there exists a $\delta_2 \succ 0$ corresponding to $\delta_1$. Note $\delta_1 \land \delta_2 \succ 0$ and we set $\delta = \delta_1 \land \delta_2$.

Since $\mathcal{A}$ is Cauchy, there exists a minimal Cauchy filter $\mathcal{L} \in \hat{X}$ such that $\mathcal{L}_\delta = \{ \mathcal{P} \in \hat{X} | \zeta(\mathcal{L}, \mathcal{P}) \leq \delta \} \in \mathcal{A}$.

We need to show that $\mathcal{L}_\delta \subseteq (\mathcal{F}_\epsilon)_\epsilon$. Let $\mathcal{P} \in \mathcal{L}_\delta$. Since $\hat{X}$ is a $V$-space, it holds that
\[
\zeta(\mathcal{F}_\epsilon, \mathcal{P}) \leq \zeta(\mathcal{F}_\epsilon, \mathcal{L}) + \zeta(\mathcal{L}, \mathcal{P}).
\]

We already know that $\zeta(\mathcal{L}, \mathcal{P}) \leq \delta \leq \delta_1$ since $\mathcal{P} \in \mathcal{L}_\delta$. So we are only left to show that $\zeta(\mathcal{F}_\epsilon, \mathcal{L}) \leq \delta_1$. By Proposition 6.2.10 we have
\[
\zeta(\mathcal{F}_\epsilon, \mathcal{L}) \leq \zeta(\mathcal{F}_\epsilon, \mathcal{F}) + \zeta(\mathcal{F}, \mathcal{L}) \leq 0 + \zeta(\mathcal{F}, \mathcal{L}),
\]
then it suffices to show
\[
\zeta(\mathcal{F}, \mathcal{L}) = \bigvee_{F \in \mathcal{F}, L \in \mathcal{L}} \tau(F, L) \leq \delta_1,
\]
which is equivalent to $\tau(F, L) \leq \delta_1$ for every $F \in \mathcal{F}$ and $L \in \mathcal{L}$ by Proposition 2.2.19(2).

Let $F_0 \in \mathcal{F}$ and $L_0 \in \mathcal{L}$. This means there exists $S \in \mathcal{A}$ such that $F_0 \in \bigcap S$, thus $F_0 \in \mathcal{G}$ for every $\mathcal{G} \in S$. Since $\mathcal{A}$ is a filter, $\mathcal{L}_\delta \cap S \in \mathcal{A}$ is non-empty.
Since \( \hat{X} \) satisfies UVA, for every \( \mathcal{G} \in \mathcal{L}_\delta \cap S \), we have

\[
\zeta(\mathcal{L}, \mathcal{G}) \leq \delta \leq \delta_2
\]

implying that \( \zeta(\mathcal{G}, \mathcal{L}) \leq \delta_1 \). Since \( F_0 \in \mathcal{G} \), we obtain

\[
\tau(F_0, L_0) \leq \bigvee_{G \in \mathcal{G}, L \in \mathcal{L}} \tau(G, L) = \zeta(\mathcal{G}, \mathcal{L}) \leq \delta_1.
\]

Since \( F_0 \in \mathcal{F} \) and \( L_0 \in \mathcal{L} \) are arbitrary, then

\[
\zeta(\mathcal{F}, \mathcal{F}) = \bigvee_{F \in \mathcal{F}, L \in \mathcal{L}} \tau(F, L) \leq \delta_1
\]

by Proposition 2.2.19(2), thus \( \zeta(\mathcal{F}_\succ, \mathcal{L}) \leq \zeta(\mathcal{F}, \mathcal{L}) \leq \delta_1 \). So we obtain

\[
\zeta(\mathcal{F}_\succ, \mathcal{P}) \leq \zeta(\mathcal{F}_\succ, \mathcal{L}) + \zeta(\mathcal{L}, \mathcal{P}) \leq \delta_1 + \delta_1 \leq \epsilon,
\]

thus \( \mathcal{P} \in (\mathcal{F}_\succ)_\epsilon \). Since \( \mathcal{P} \in \mathcal{L}_\delta \) is arbitrary, it follows that \( \mathcal{L}_\delta \subseteq (\mathcal{F}_\succ)_\epsilon \) and since \( \mathcal{L}_\delta \in \mathcal{A} \), we obtain \( (\mathcal{F}_\succ)_\epsilon \in \mathcal{A} \). Then since \( \epsilon > 0 \) is arbitrary, it follows that \( \mathcal{A} \) converges to \( \mathcal{F}_\succ \).

\[ \square \]

**Theorem 6.4.18.** Let \( X \) be a \( V \)-space satisfying UVA. Then there exists a bijective isometry between \( \hat{X} \) and \( \check{X}_0 \).

**Proof.** We show the proof in two parts. Firstly, we show there exists a unique minimal Cauchy filter in each equivalence class \([\mathcal{F}]\) in \( \hat{X}_0 \). For the second part, we define a function \( \psi : \hat{X} \to \check{X}_0 \) and verify that it is a bijective isometry.

1. For \( \check{X}_0 \), each element is an equivalence class \([\mathcal{F}]\) of Cauchy filters. We will verify that each equivalence class in \( \check{X}_0 \) has a unique minimal Cauchy filter \( \mathcal{F}_\succ \).

   (a) Existence of a minimal Cauchy filter in \([\mathcal{F}]\): By Corollary 6.4.5, since \( X \) satisfies UVA and \( \mathcal{F} \) is Cauchy, \( \mathcal{F}_\succ \) is a minimal Cauchy filter. We will show that \( \mathcal{F}_\succ \in [\mathcal{F}] \). Since \( \mathcal{F}_\succ \subseteq \mathcal{F} \),

   \[
   \zeta(\mathcal{F}, \mathcal{F}_\succ) = 0 = \zeta(\mathcal{F}_\succ, \mathcal{F}),
   \]

   thus \( \mathcal{F}_\succ \in [\mathcal{F}] \).
(b) Uniqueness of the minimal Cauchy filter in \([\mathcal{F}]\): Suppose \(\mathcal{F}_1, \mathcal{F}_2 \in [\mathcal{F}]\) are minimal Cauchy filters. So

\[
\zeta(\mathcal{F}_1, \mathcal{F}_2) = 0 = \zeta(\mathcal{F}_2, \mathcal{F}_1),
\]

which implies that \(\mathcal{F}_1 \cap \mathcal{F}_2\) is Cauchy by the argument in Proposition 6.4.16. Since \(\mathcal{F}_1, \mathcal{F}_2\) are minimal Cauchy filters, then

\[
\mathcal{F}_1 = \mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}_2.
\]

2. Now, we define the mapping \(\psi: \hat{X} \to \hat{X}_0\) by assigning the minimal Cauchy filter \(\mathcal{F}_\succ \in \hat{X}\) to the equivalence class \([\mathcal{F}]\) where \(\mathcal{F}_\succ \in [\mathcal{F}]\) in \(\hat{X}\). We are left to show that \(\psi\) is injective, surjective and distance preserving.

(a) \(\psi\) is injective: Suppose \(\psi(\mathcal{F}_\succ) = [\mathcal{F}] = [\mathcal{G}] = \psi(\mathcal{G}_\succ)\). This means for \(\mathcal{F}_\succ \in [\mathcal{F}]\) and \(\mathcal{G}_\succ \in [\mathcal{G}]\). Since \([\mathcal{F}] = [\mathcal{G}]\), then they have a unique minimal Cauchy filter, thus \(\mathcal{F}_\succ = \mathcal{G}_\succ\).

(b) \(\psi\) is surjective: Let \([\mathcal{F}] \in \hat{X}_0\). Then there exists a unique minimal Cauchy filter \(\mathcal{F}_\succ \in [\mathcal{F}]\). This means \(\mathcal{F}_\succ \in \hat{X}\), thus we have \(\psi(\mathcal{F}_\succ) = [\mathcal{F}]\).

(c) \(\psi\) is distance preserving: We will show that the mapping is distance preserving, i.e.,

\[
\zeta(\psi(\mathcal{F}), \psi(\mathcal{G})) = \zeta(\mathcal{F}, \mathcal{G}).
\]

Since

\[
\zeta([\mathcal{F}], [\mathcal{G}]) = \zeta(\mathcal{F}, \mathcal{G}),
\]

it suffices to show that

\[
\zeta(\mathcal{F}_\succ, \mathcal{G}_\succ) \leq \zeta(\mathcal{F}, \mathcal{G})
\]

and

\[
\zeta(\mathcal{F}_\succ, \mathcal{G}_\succ) \geq \zeta(\mathcal{F}, \mathcal{G}).
\]

Using Triangle Inequality, we can easily see that

\[
\zeta(\mathcal{F}_\succ, \mathcal{G}_\succ) \leq \zeta(\mathcal{F}_\succ, \mathcal{F}) + \zeta(\mathcal{F}, \mathcal{G}) + \zeta(\mathcal{G}, \mathcal{G}_\succ) \\
\leq 0 + \zeta(\mathcal{F}, \mathcal{G}) + 0 \\
\leq \zeta(\mathcal{F}, \mathcal{G})
\]
and
\[
\zeta(\mathcal{F}, \mathcal{G}) \leq \zeta(\mathcal{F}, \mathcal{F}') + \zeta(\mathcal{F}', \mathcal{G}') + \zeta(\mathcal{G}', \mathcal{G}) \\
\leq 0 + \zeta(\mathcal{F}', \mathcal{G}') + 0 \\
\leq \zeta(\mathcal{F}', \mathcal{G}')
\]
as required.

**Definition 6.4.19.** Let $X$ be a $V$-space satisfying UVA. The function $\iota: X \to \hat{X}$ given by $\iota(x) = \mathcal{F}_x$ is called the canonical embedding.

**Remark 6.4.20.** Note that the canonical embedding $\iota$ is injective, if and only if, $X$ is separated.

**Lemma 6.4.21.** Let $X$ be a $V$-space satisfying UVA. Then the canonical embedding $\iota$ is an isometry.

**Proof.** Suppose $X$ is UVA. By Proposition 6.2.10, $\zeta(\mathcal{F}_x, \mathcal{F}_y) = \zeta(\mathcal{B}_x, \mathcal{B}_y)$, so we need to show that $\zeta(\mathcal{B}_x, \mathcal{B}_y) = d(x, y)$, or equivalently,
\[
\zeta(\mathcal{B}_x, \mathcal{B}_y) \leq d(x, y)
\]
and
\[
\zeta(\mathcal{B}_x, \mathcal{B}_y) \geq d(x, y).
\]
For the first inequality, by Proposition 2.2.19(2), it suffices to show that
\[
\tau(x, y) = \bigwedge_{a \in x, b \in y} d(a, b) \leq d(x, y)
\]
for every $\epsilon > 0$ and $\epsilon' > 0$ which indeed holds since $x \in x_\epsilon$ and $y \in y_{\epsilon'}$.

For the second inequality, since $X$ satisfies UVA, by Proposition 6.4.9, we have $\mathcal{F}_y = \mathcal{F}_y'$, so it suffices to show that $d(x, y) \leq \zeta(\mathcal{B}_x, \mathcal{B}_y')$. Let $\rho > 0$. Then by Lemma 2.3.19, there exists $\eta > 0$ such that $\eta + \eta \leq \rho$. By Proposition 6.1.11, there is a $\delta > 0$ such that
\[
\tau(x, y) \geq d(x, y) - 2\eta \\
= d(x, y) - \rho
\]
and thus $d(x, y) \geq \zeta(\mathcal{B}_x, \mathcal{B}_y') + \rho$. Since $\rho > 0$ is arbitrary, the desired inequality follows.
Theorem 6.4.22. Let $X$ be a separated $V$-space which satisfies UVA. Then $\iota(X)$ is dense in $X$.

Proof. Let $\epsilon > 0$ and $G \in \hat{X}$. By UVA property, there exists $\delta > 0$ corresponding to $\epsilon$. We need to show that there exists $x \in X$ such that $\iota(x) = \mathcal{F}_x \in \mathcal{G}$ ($\mathcal{G}$ is a ball of minimal Cauchy filters around $G$ with radius $\epsilon$), which is equivalent to $\zeta(G, \mathcal{F}_x) \leq \epsilon$.

By Proposition 6.2.10, we have $\zeta(G, \mathcal{F}_x) = \zeta(G, \mathcal{B}_x)$ and since $\mathcal{B}_x = \{x_\eta \mid \eta > 0\}$, it suffices to show that

$$\zeta(G, \mathcal{B}_x) = \bigvee_{G \in \mathcal{G}, \eta > 0} \tau(G, x_\eta) \leq \epsilon$$

which is equivalent to $\tau(G, x_\eta) \leq \epsilon$ for every $G \in \mathcal{G}$ and $\eta > 0$. Since $\mathcal{G}$ is Cauchy, there exists an $x \in X$ such that $x_\delta \in \mathcal{G}$.

Let $G_0 \in \mathcal{G}$ and $\eta > 0$. We know that $G_0 \cap x_\delta \in \mathcal{G}$. Then by Proposition 6.1.6(3), it holds that

$$\tau(G_0, x_\eta) \leq \tau(G_0 \cap x_\delta, x_\eta) \leq \tau(G_0 \cap x_\delta, x).$$

For every $y \in G_0 \cap x_\delta$, it holds that $d(x, y) \leq \delta$ which implies that $d(y, x) \leq \epsilon$, then

$$\tau(G_0 \cap x_\delta, x) = \bigwedge_{y' \in G_0 \cap x_\delta} d(y', x) \leq d(y, x) \leq \epsilon,$$

thus $\tau(G_0, x_\eta) \leq \epsilon$. Since $G_0 \in \mathcal{G}$ and $\eta > 0$ are arbitrary, it follows that

$$\zeta(G, \mathcal{B}_x) \leq \epsilon$$

which completes the proof.

Theorem 6.4.23. The usual universal property holds for the completion. In other words, for every uniformly continuous $f : X \to Y$ where $X, Y$ are separated $V$-spaces and $Y$ is complete, there exists a unique function $F : \hat{X} \to Y$ such that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\iota \downarrow & & \downarrow F \\
\hat{X} & \xrightarrow{F} & Y
\end{array}$$

commutes.

Proof. Let $f : X \to Y$ be a uniformly continuous function and $G \in \hat{X}$. The proof is


CHAPTER 6. CAUCHY COMPLETION

divided into six parts. Through $f$ and $\mathcal{G}$, we first verify the collection

$$\mathcal{B} = \{ f(G) \mid G \in \mathcal{G} \}$$

is a filter base. Then using $\mathcal{B}$, we construct a filter $\mathcal{H}$ on $Y$ and show that $\mathcal{H}$ is Cauchy. For the next part, since $Y$ is complete, $\mathcal{H}$ converges to some point $y \in Y$, so we define $F$ by $F(\mathcal{G}) = y$ and show that it is a function. For the fourth part, we verify that $F(\mathcal{F}_x) = f(x)$ and then in the fifth part, we show that $F$ is also uniformly continuous. Then the last part shows that $F$ is unique.

1. We first construct a Cauchy filter in $Y$ using $\mathcal{G}$ and $f$. We have a collection $\mathcal{B} = f(\mathcal{G}) = \{ f(G) \mid G \in \mathcal{G} \}$ which we verify is a filter base on $Y$.

(a) Non-empty: Since $X \in \mathcal{G}$, then the range of $f$, $f(X) \in \mathcal{B}$, thus $\mathcal{B}$ is non-empty.

(b) $\emptyset \notin \mathcal{B}$: For every $B \in \mathcal{B}$, there exists a $G \in \mathcal{G}$ such that $f(G) = B$ and $G \neq \emptyset$. Then there is at least an element $g \in G$, then $f(g) \in B$, thus $B \neq \emptyset$.

(c) For every $R, R' \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $C \subseteq R \cap R'$: Let $R, R' \in \mathcal{B}$. This means there exists $G, G' \in \mathcal{G}$ such that $f(G) = R$ and $f(G') = R'$. Then $G \cap G' \in \mathcal{G}$, so $f(G \cap G') \in \mathcal{B}$. It suffices to show that

$$f(G \cap G') \subseteq f(G) \cap f(G').$$

Let $b \in f(G \cap G')$. This means there exists an $a \in G \cap G'$ such that $f(a) = b$. Since $a \in G$ and $a \in G'$, we have $f(a) \in f(G)$ and $f(a) \in f(G')$ thus

$$f(a) = b \in f(G) \cap f(G').$$

Since $b \in f(G \cap G')$ is arbitrary, it follows that $f(G \cap G') \subseteq f(G) \cap f(G')$.

2. We generate a filter on $Y$, denoted by $\mathcal{H}$, using the filter base $\mathcal{B}$. Next, we show that $\mathcal{H}$ is Cauchy.

Let $\epsilon > 0$. Since $f$ is uniformly continuous, there exists a $\delta > 0$ corresponding to $\epsilon$. Since $\mathcal{G}$ is Cauchy, there exists an $x \in X$ such that $x_\delta \in \mathcal{G}$. So $f(x_\delta) \in \mathcal{H}$, then it suffices to show that

$$f(x_\delta) \subseteq f(x).$$

Let $z \in f(x_\delta)$. This means there is a $y \in x_\delta$ such that $f(y) = z$. Since $y \in x_\delta$,
we have \( d(x, y) \leq \delta \) which implies that
\[
d(f(x), f(y)) \leq \epsilon,
\]
thus \( z = f(y) \in f(x) \). Since \( z \in f(x) \) is arbitrary, it follows that
\[
f(x) \subseteq f(x) \epsilon.
\]
thus \( f(x) \in \mathcal{H} \). Since \( \epsilon > 0 \) is arbitrary, we conclude that \( \mathcal{H} \) is Cauchy.

3. Since \( Y \) is complete, \( \mathcal{H} \) converges to an element \( y \in Y \). We verify first that the limit of \( \mathcal{H} \) is unique. Let \( a \) and \( b \) be two limits of \( \mathcal{H} \). Let \( \epsilon > 0 \). Then there is a \( \delta > 0 \) such that \( d(a, b) \leq \tau(a, b) + 2 \cdot \epsilon \). Since \( a, b \in \mathcal{H} \), we have \( \tau(a, b) \leq \zeta(\mathcal{H}, \mathcal{H}) = 0 \) and so \( d(a, b) \leq 2 \cdot \epsilon \). As \( \epsilon > 0 \) was arbitrary, we have \( d(a, b) = 0 \). Similarly, \( d(b, a) = 0 \). Since \( Y \) is separated, \( a = b \).

Now we define a relation \( F: \hat{X} \to Y \) by assigning \( G \in \hat{X} \) to the element \( y \in Y \).

We now verify that \( F \) is a function. Suppose \( F, G \in \hat{X} \) such that \( F = G \).

For \( F \), we generate a Cauchy filter \( \mathcal{H}_1 \) which converges to some point \( y_1 \in Y \) and for \( G \), we generate a Cauchy filter \( \mathcal{H}_2 \) on \( Y \) which converges to some point \( y_2 \in Y \). We need to show that
\[
F(\mathcal{F}) = y_1 = y_2 = F(\mathcal{G}).
\]

Since it holds that \( \mathcal{F} \subseteq \mathcal{G} \), then
\[
B_1 = f(\mathcal{F}) \subseteq f(\mathcal{G}) = B_2,
\]
thus \( \mathcal{H}_1 \subseteq \mathcal{H}_2 \). Then since \( \mathcal{H}_1 \to y_1 \), we have \( \mathcal{H}_2 \to y_1 \), thus \( y_2 = y_1 \).

4. We show that the diagram commutes, i.e., \( F \circ \iota(x) = F(\mathcal{F}_x) = f(x) \). Let \( x \in X \). Applying \( \iota \), we have the minimal Cauchy filter \( \iota(x) = \mathcal{F}_x \) in \( \hat{X} \). Using \( \mathcal{F}_x \) and the construction above, we have a Cauchy filter \( \mathcal{H} \) on \( Y \). We need to show that \( \mathcal{H} \) converges to \( f(x) \). Let \( \epsilon > 0 \). Since \( f \) is uniformly continuous, there exists a \( \delta > 0 \) corresponding to \( \epsilon \). We have \( x_\delta \in \mathcal{F}_x \), then \( f(x_\delta) \in \mathcal{H} \), so it suffices to show that
\[
f(x_\delta) \subseteq f(x) \epsilon.
\]
The argument for this inclusion is already given in part (2). Thus we have \( f(x)_\epsilon \in \mathcal{H} \) and since \( \epsilon > 0 \) is arbitrary, \( \mathcal{H} \) converges to \( f(x) \). This means that
\[
F \circ \iota(x) = F(\iota(x)) = F(\mathcal{F}_x) = f(x).
\]
5. For the last part, we show that $F$ is uniformly continuous: Let $\epsilon > 0$. For $\mathcal{G}, \mathcal{G}' \subseteq \mathcal{X}$, we have $F(\mathcal{G}) = y$ and $F(\mathcal{G}') = y'$ where $\mathcal{H}$ and $\mathcal{H}'$ are constructed using $\mathcal{G}$ and $\mathcal{G}'$, respectively and $\mathcal{H} \to y$ and $\mathcal{H}' \to y'$. We need to show that there exists a $\delta > 0$ such that $\zeta(\mathcal{G}, \mathcal{G}') \leq \delta$ implies that

$$d(y, y') \leq \epsilon.$$ 

By Proposition 2.3.18, there exists a $\delta_1 > 0$ such that $3 \cdot \delta_1 \leq \epsilon$. By UVA, there exists $\delta_2 > 0$ corresponding to $\delta_1$ and since $f$ is uniformly continuous, there exists a $\delta_3 > 0$ corresponding to $\delta_2$. By Proposition 2.3.18, there exists $\delta_4 > 0$ such that $5 \cdot \delta_4 \leq \delta_3$ and by UVA property, there is a $\delta_5 > 0$ corresponding to $\delta_4$. Note that $\delta_1 \wedge \delta_5 > 0$ so set $\delta = \delta_1 \wedge \delta_5$.

Suppose

$$\zeta(\mathcal{G}, \mathcal{G}') = \bigvee_{G \in \mathcal{G}, G' \in \mathcal{G}'} \tau(G, G') \leq \delta_4.$$ 

By Proposition 2.2.19(2), this is equivalent to $\tau(G, G') \leq \delta_4$ for every $G \in \mathcal{G}$ and $G' \in \mathcal{G}'$. Since $\mathcal{G}$ and $\mathcal{G}'$ are Cauchy, there exists $x, x' \in X$ such that $x_\delta \in \mathcal{G}$ and $x'_\delta \in \mathcal{G}'$, and $\tau(x_\delta, x'_\delta) \leq \delta_4$. Note that for every $a \in x_\delta$ and $a' \in x'_\delta$, $d(x, a) \leq \delta \leq \delta_5$ implies $d(a, x) \leq \delta_4$ and $d(a', x') \leq \delta \leq \delta_4$. Then by Proposition 6.1.11

$$d(a, a') \leq d(a, x) + d(x, x') + d(x', a') \leq \delta_4 + [\tau(x_\delta, x'_\delta) + 2 \cdot \delta_4] + \delta_4 \leq 5 \cdot \delta_4 \leq \delta_3.$$ 

Also, since $\mathcal{H} \to y$ and $\mathcal{H}' \to y'$, we have $y_\delta \in \mathcal{H}'$ and $y'_\delta \in \mathcal{H}'$. We know that $f(x_\delta) \in \mathcal{H}$ and $f(x'_\delta) \in \mathcal{H}'$ then $y_\delta \cap f(x_\delta) \in \mathcal{H}$ and $y'_\delta \cap f(x'_\delta) \in \mathcal{H}'$. Let $k \in y_\delta \cap f(x_\delta)$ and $k' \in y'_\delta \cap f(x'_\delta)$. We see that $k \in f(x_\delta)$ and $k' \in f(x'_\delta)$ so there exists $a \in x_\delta$ and $a' \in x'_\delta$ such that $f(a) = k$ and $f(a') = k'$. Then it holds that $d(a, a') \leq \delta_3$ implying that

$$d(k, k') \leq \delta_1$$

by $f$. Also, we see that

$$d(y, k) \leq \delta_1$$

and $d(y', k') \leq \delta_2$ which implies

$$d(k', y') \leq \delta_1$$


by UVA. By Triangle Inequality, we obtain
\[ d(y, y') \leq d(y, k) + d(k, k') + d(k', y') \leq 3 \cdot \delta_1 \leq \epsilon. \]

Since \( \epsilon > 0 \) is arbitrary, we conclude that \( F \) is uniformly continuous.

6. Suppose \( F': \hat{X} \to Y \) is another uniformly continuous function that makes the diagram commute. Let \( \mathcal{F} \in \hat{X} \). Let \( \epsilon > 0 \). Then there is a \( \epsilon_2 > 0 \) such that \( 2 \cdot \epsilon_2 \leq \epsilon \) by Proposition 2.3.19. Since \( F \) (resp. \( F' \)) is uniformly continuous, there is a \( \delta_1 > 0 \) (resp. \( \delta_2 > 0 \)) corresponding to \( \epsilon_2 \) in the definition of uniform continuity. Let \( \delta_3 = \delta_1 \wedge \delta_2 > 0 \). Since \( \hat{X} \) satisfies UVA, there exists \( \delta_4 > 0 \) corresponding to \( \delta_3 \) in the definition of UVA. Let \( \delta = \delta_3 \wedge \delta_4 > 0 \). Since \( \iota(X) \) is dense in \( \hat{X} \), there is a \( x \in X \) such that \( \varsigma(\mathcal{F}, B_x) \leq \delta \), hence \( d_Y(F(\mathcal{F}), F(B_x)) \leq \epsilon_2 \) and \( d_Y(F'(\mathcal{F}), F'(B_x)) \leq \epsilon_2 \). But \( F(B_x) = f(x) = F'(B_x) \). Therefore
\[
\begin{align*}
\quad d_Y(F(\mathcal{F}), F'(\mathcal{F})) &\leq d_Y(F(\mathcal{F}), f(x)) + d_Y(f(x), F'(\mathcal{F})) \\
&\leq 2 \cdot \epsilon_2 \leq \epsilon.
\end{align*}
\]

Since \( \epsilon > 0 \) was arbitrary, we conclude that \( d_Y(F(\mathcal{F}), F'(\mathcal{F})) = 0 \). Similarly, \( d_Y(F'(\mathcal{F}), F(\mathcal{F})) = 0 \). As \( Y \) is a separated space, we must have \( F(\mathcal{F}) = F'(\mathcal{F}) \forall \mathcal{F} \in \hat{X} \) and thus \( F = F' \).

\[ \blacksquare \]

**Theorem 6.4.24.** Let \( X, Y \) be separated \( V \)-spaces and \( \iota: X \to \hat{X} \) an isometric embedding which is dense in \( \hat{X} \). Then if \( F_1, F_2: \hat{X} \to Y \) are such that \( F_1 \circ \iota = F_2 \circ \iota \), then \( F_1 = F_2 \).
Chapter 7

Future Research Projects

Since $V$-spaces were introduced a bit more than 20 years ago, the theory can hardly be considered as extensive, compared with metric spaces. This chapter will focus on the research prospects that arise from this thesis. In the previous chapter, we constructed a completion using filters and since it is a well-known result that filters and nets are equivalent notions, it would be interesting to consider a completion construction using nets. To add on the subject, it was noticed during the literature review that since quasi-metrics do not satisfy symmetry, different completion constructions arise, such as double completion [28], $B$-completion [29], Smyth completion [12], bi-completion, half-completion [1], and so on, a project could easily be formed encompassing the study of these completion constructions with respect to $V$-spaces. In addition, in Topology, there is a standard notion of compactness, thus an interesting project would be to define a relevant notion of compactness in $V$-spaces and show whether it is possible to present a compactification construction for it.

A natural question to ask is given value quantales $V$ and $W$, when is $\text{Met}_V$ equivalent to $\text{Met}_W$.

When comparing the completion constructions of metric (or uniform) spaces to quasi-metric (or quasi-uniform) spaces with a category-theoretic perspective, we noticed that the discrepancies may be be explained with a notion similar to fibrations. Dr. Weiss and I are currently working on introducing a concept of a $B$-controlled weighted category which would explain this discrepancy efficiently.

Moreover, in Section 5.2, we define and verify the addition operation on the value distributive lattice $\Delta$. This operation turns out to endow $\Delta_{st}$ with interesting properties, similar to $[0, \infty]$. Surprisingly, such a detailed account of $\Delta$ is hard to find in the literature. Further research could be directed at analyzing the structure of $\Delta_{st}$. 
Furthermore, Flagg used value quantales to define V-spaces, as a generalization of metric spaces, but it need not stop there. Value quantales can be placed in other fields of mathematics to provide a generalization to existing theories. For instance, in Algebraic Topology, we have these basic existing notions: Given paths in topological space $X \gamma_1: [0, 1] \to X$ and $\gamma_2: [0, 1] \to X$, a homotopy between paths is a continuous function $h: [0, 1] \times [0, 1] \to X$ such that

1. $h(a, b) = \gamma_1(a)$ for fixed $b \in [0, 1]$ and
2. $h(a, b) = \gamma_2(b)$ for fixed $a \in [0, 1]$.

Since $[0, 1] \subseteq [0, \infty]$, it would be an interesting research project to define a suitable notion of subset $W \subseteq V$ and a notion of product $\times$ such that we have paths $\gamma: W \to X$ and a $V$-homotopy between two paths $W \times W \to X$. It is natural to consider the fundamental group construction in this setting.

In addition, value quantales could also be used to generalize normed vector spaces. A norm on a vector space $A$ is a function $\|\cdot\|: A \to [0, \infty]$ which satisfies

1. Triangle Inequality: $\|P + Q\| \leq \|P\| + \|Q\|$
2. Homogeneity: $\|\alpha P\| = |\alpha| \cdot \|P\|$.

We can easily see that the definition of a norm is grounded in $[0, \infty]$ with the standard metric. So a natural question to ask is what happens when $[0, \infty]$ is replaced by $V$. For the second property, we would need to define a sensible notion of an action of $[0, \infty]$ on $V$, similar in principle, to a group action found in Group Theory.
Bibliography


